Math 160, Test #3B

Solutions

1. Find the derivative (with respect to x) of each of the following functions.

   (a) $\dfrac{1}{(6x+3)^4}$

   Solution: Rewrite the function as $(6x+3)^{-3}$, and then use the chain rule:

$$\begin{aligned}\frac{d}{dx}((6x+3)^{-4}) &= (-4)(6x+3)^{-5}\cdot\frac{d}{dx}(6x+3)\\ &= -4(6x+3)^{-5}\cdot 6\\ &= -24(6x+3)^{-5}\\ &= \frac{-24}{(6x+3)^5}\end{aligned}$$

   (b) $\sqrt{6x^3-3x^4}$

   Solution: Rewrite the function as $(6x^3-3x^4)^{1/2}$, and then use the chain rule:

$$\begin{aligned}\frac{d}{dx}(6x^3-3x^4)^{1/2}) &= \frac{1}{2}(6x^3-3x^4)^{-1/2}\cdot\frac{d}{dx}((6x^3-3x^4)\\ &= \frac{1}{2}(6x^3-3x^4)^{-1/2}\cdot(18x^2-12x^3)\\ &= \frac{9x^2-6x^3}{2\sqrt{6x^3-3x^4}}\\ &= \frac{3x^2(3-2x)}{2\sqrt{6x^3-3x^4}}\end{aligned}$$

2. For this problem, show all necessary work; do *not* use your graphing calculator.

Let $f(x) = 6x^3 - 3x^4$.

(a) Find the critical numbers of f.

Solution: The critical numbers of $f(x)$ are all values of x for which $f'(x) = 0$:

$$\begin{aligned} f'(x) &= 0 \\ 18x^2 - 12x^3 &= 0 \\ 6x^2(3 - 2x) &= 0 \\ x &= 0 \text{ or } x = 3/2 \end{aligned}$$

So the critical numbers are $x = 0$ and $x = 3/2$

(b) Find the x-values of the relative maxima and minima (if any) of f. Solution: The graph of f may have relative extrema and $x = 0$ and/or at $x = 3/2$. We can test the sign of f' on the intervals determined by these numbers to figure out where f is increasing and where f is decreasing.

- On the interval $(-\infty, 0)$, we can use test point $x = -1$:

  $$f'(-1) = 18 - (-12) = 30 > 0.$$

  Therefore, $f'(x)$ is positive on $(-\infty, 0)$, so f is increasing on this interval.
- On the interval $(0, 3/2)$, we can use test point $x = 1$:

  $$f'(1) = 18 - 12 = 6 > 0.$$

  Therefore, $f'(x)$ is positive on $(0, 3/2)$, so f is increasing on this interval.
- On the interval $(3/2, \infty)$, we can use test point $x = 2$:

  $$f'(2) = 72 - 96 = -24 < 0.$$

  Therefore, $f'(x)$ is negative on $(3/2, \infty)$, so f is decreasing on this interval.

Conclusions:

- Since f is increasing on both $(-\infty, 0)$ and on $(0, 3/2)$, there is no relative extreme point at $x = 0$.
- Since f is increasing on $(0, 3/2)$ and decreasing on $(3/2, \infty)$, f has a relative maximum at $x = 3/2$.

Relative maxima: $x = 3/2$

Relative minima: DNE

3. For this problem, show all necessary work; do *not* use your graphing calculator.

Let $g(x) = x^2 - \dfrac{6}{x}$.

(a) Find the inflection point of the graph of $y = g(x)$. Find the *exact* x- and y-coordinates. (Comment: the wording of the problem implicitly hints that you should find exactly one inflection point.)

To find inflection points, we must first find $g''(x)$:

$$\begin{aligned} g(x) &= x^2 - 6x^{-1} \\ g'(x) &= 2x + 6x^{-2} \\ g''(x) &= 2 - 12x^{-3} \\ &= 2 - \frac{12}{x^3} \end{aligned}$$

Inflection points are points in the domain of g at which the concavity changes; these may occur where $g'(x) = 0$ or $g'(x)$ does not exist. Since $x = 0$ is not in the domain of g, we can disregard that possibility; all that remains is the value(s) of x at which $g''(x) = 0$:

$$\begin{aligned} g''(x) &= 0 \\ 2 - \frac{12}{x^3} &= 0 \\ 2x^3 - 12 &= 0 \\ 2x^3 &= 12 \\ x^3 &= 12 \\ x &= \sqrt[3]{6} \end{aligned}$$

So, if $g''(x)$ changes sign at $x = \sqrt[3]{6}$, then g has an inflection point at $x = \sqrt[3]{6}$.

To determine the sign of $g''(x)$, we can select test points for each of the intervals $(-\infty, 0)$, $(0, \sqrt[3]{6})$, and $(\sqrt[3]{6}, \infty)$. (Note: for this part of the problem, we don't really need to know what happens when $x < 0$; however, we'll need this information for parts (b) and (c), so we may as well find it now.)

- On the interval $(-\infty, 0)$, we can use test point $x = -1$: $g''(-1) = 2-(-12) = 14 > 0$. Therefore, $g''(x)$ is positive on $(-\infty, 0)$, so g is concave up on this interval.
- On the interval $(0, \sqrt[3]{6})$, we can use test point $x = 1$: $g''(1) = 2 - 12 = -10 < 0$. Therefore, $g''(x)$ is negative on $(0, \sqrt[3]{6})$, so g is concave down on this interval.
- On the interval $(\sqrt[3]{6}, \infty)$, we can use test point $x = 2$: $g''(3) = 2 - \frac{12}{8} = \frac{1}{2} > 0$. Therefore, $g''(x)$ is positive on $(\sqrt[3]{6}, \infty)$, so g is concave up on this interval.

We've shown that g changes concavity at $x = \sqrt[3]{6}$; therefore, this is an inflection point of the graph of g. The y-coordinate of this point is

$$g(\sqrt[3]{6}) = g(6^{1/3}) = (6^{1/3})^2 - \frac{6}{4^{1/3}} = 6^{2/3} - 6^{2/3} = 0.$$

So, the inflection point's *exact* coordinates are $(\sqrt[3]{6}, 0)$.

(b) For what values of x is the graph of $g(x)$ concave up? (Write your answer using interval notation.)

(c) For what values of x is the graph of $g(x)$ concave down? (Write your answer using interval notation.)

Answers: As shown in the work for part (a) above, g is concave up on the intervals $(-\infty, 0)$ and $(\sqrt[3]{6}, \infty)$, and g is concave down on the interval $(0, \sqrt[3]{6})$.

4. Recall that elasticity of demand is defined as

$$E(p) = \frac{-pf'(p)}{f(p)},$$

where $x = f(p)$ is the demand function.

For the demand function

$$x = 18 - \frac{p^2}{50},$$

compute the elasticity of demand, and use your result to answer each of the following questions.

(a) Is demand elastic or inelastic when $p = 20$?

Solution: First, we need to find $E(p)$. Note that $f(p) = 18 - \frac{p^2}{50}$, which implies $f'(p) = -\frac{p}{25}$. Therefore,

$$\begin{aligned} E(p) &= \frac{-pf'(p)}{f(p)} \\ &= \frac{-p(-p/25)}{18 - \frac{p^2}{50}} \\ &= \frac{p^2/25}{18 - p^2/50} \cdot \frac{50}{50} \\ &= \frac{2p^2}{900 - p^2} \end{aligned}$$

So,

$$E(20) = \frac{2(20)^2}{900 - 20^2} = \frac{2 \cdot 400}{900 - 400} = \frac{800}{500} = \frac{8}{5} > 1.$$

Since $E(20) > 1$, demand is elastic when $p = 20$.

(b) For what value of p is demand unitary? Give an exact answer, and also a decimal answer rounded to the nearest cent.

Solution: Demand is unitary when $E(p) = 1$:

$$\begin{aligned} E(p) &= 1 \\ \frac{2p^2}{900 - p^2} &= 1 \\ 2p^2 &= 900 - p^2 \\ 3p^2 &= 900 \\ p^2 &= 300 \\ p &= \sqrt{300} \approx 17.32 \end{aligned}$$

Therefore, demand is unitary when the price is \$17.32.

5. Suppose the weekly demand for burritos is given by $p = 7.8 - 0.06x$, where p denotes price per burrito (in dollars) and x is the number of burritos demanded per week at price p.

   (a) Find the revenue function, R. (Hint: recall that revenue is price per unit times number of units sold.) Solution: $R(x) = xp(x) = 7.8x - 0.06x^2$ (Note: the units of R are dollars)

   (b) Find the marginal revenue function, R'. Solution: $R'(x) = 7.8 - 0.12x$ (Note: The units of R' are dollars per burrito)

   (c) Find the value of $R'(70)$, and interpret the result. (Write at least one sentence, demonstrating that you understand what "marginal revenue" measures.)

   Solution: $R'(110) = 7.8 - 0.12(70) = 7.8 - 8.4 = -0.6$

   Interpretation: Marginal revenue is the addition revenue we expect to earn if we sell one more burrito. This result tells us that, if we are selling 70 burritos (at our current price per burrito), then total revenue would actually *decrease* by \$0.60 (or, 60 cents) for each additional burrito that we would sell (as a result of lowering the price per burrito).

6. For this problem, sketch a graph of the function:

$$f(x) = 600x^3 - 300x^4$$

Solution: Here is the work for finding the important points on the graph:

- Intercepts:

$$\begin{aligned} 600x^3 - 300x^4 &= 0 \\ 300x^3(2 - x) &= 0 \\ x = 0,\ x &= 2 \end{aligned}$$

- Potential relative extrema: Set $f'(x) = 0$:

$$\begin{aligned} 1800x^2 - 1200x^3 &= 0 \\ 600x^2(3 - 2x) &= 0 \\ x = 0,\ x &= 3/2 \end{aligned}$$

- Potential inflection points: Set $f''(x) = 0$:

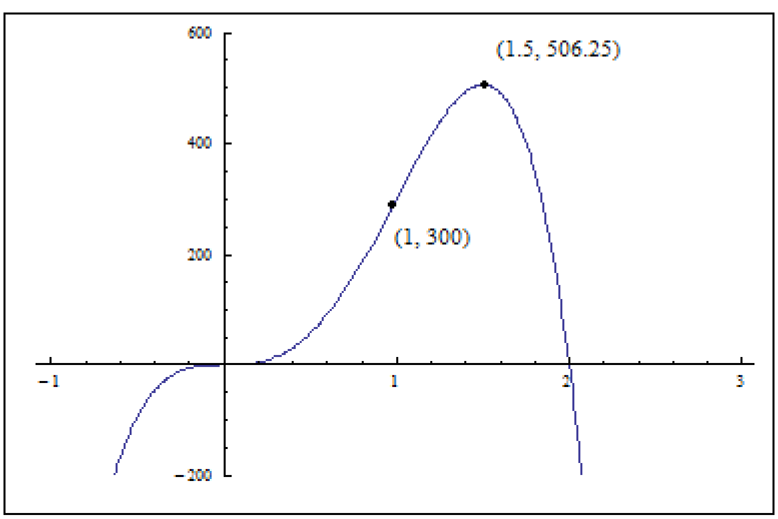
$$\begin{aligned} 3600x - 3600x^2 &= 0 \\ 3600x(1 - x) &= 0 \\ x = 0,\ x &= 1 \end{aligned}$$

The points on the graph corresponding to these x-values are as follows:

$$(0, 0), (2, 0), (1.5, 506.25), (1, 300)$$

Based on these results, we should graph $y = f(x)$ on an window that includes (at least) an x range of 0 to 2 and a y range of 0 to 506.25.

In the diagram shown below, we've graphed $y = f(x)$ using a viewing window of $-1 \leq x \leq 3$, $-200 \leq y \leq 600$.



As we can see, there is a relative maximum at (1.5, 506.25) and an inflection point at (1, 300).