Congruence Properties of Partitions: Part 2

Continuing our journey of partition congruences from the last post we now prove the congruences modulo 7 and 11.

Proof of $p(7n+5)\equiv 0\pmod{7}$

Both the proofs are similar to those given for the congruence modulo 5.

First Proof: Let $f(q)=(1-q)(1-q^2)(1-q^3)\cdots$. Arguing like the case of modulo 5 we have:

$$\begin{aligned}q^2\{f(q)\}^6&=q^2\left\{\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}\right\}^2\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}(2m+1)(2n+1)q^{m(m+1)/2+n(n+1)/2+2}\end{aligned}$$

We analyze the powers of q on RHS which are multiples of 7 and then we find that

$$\begin{aligned}\frac{m(m+1)}{2}+\frac{n(n+1)}{2}+2&\equiv 0\pmod{7}\\\Leftrightarrow 4m(m+1)+4n(n+1)+16&\equiv 0\pmod{7}\\\Leftrightarrow 4m(m+1)+4n(n+1)+2&\equiv 0\pmod{7}\\\Leftrightarrow (2m+1)^2+(2n+1)^2&\equiv 0\pmod{7}\end{aligned}$$

Both $(2m+1)^2$ and $(2n+1)^2$ take the values $0,1,2,4$ modulo 7. The only way the above congruence can be satisfied is when both $(2m+1)$ and $(2n+1)$ are divisible by 7. Thus the coefficient of q^{7n} in $q^2\{f(q)\}^6$ is divisible by 49 and hence by 7.

Now we have

$$\begin{aligned}(1-q)^7&\equiv 1-q^7\pmod{7}\\\Rightarrow\frac{1}{(1-q)^7}&\equiv\frac{1}{1-q^7}\pmod{7}\\\Rightarrow\frac{1-q^7}{(1-q)^7}&\equiv 1\pmod{7}\end{aligned}$$

Replacing q by $q^2,q^3,\ldots$ in the above equation and multiplying the resulting equations we get

$$\frac{f(q^7)}{\{f(q)\}^7}\equiv 1\pmod{7}$$

Again we can write

$$\frac{q^2f(q^7)}{f(q)}=q^2\{f(q)\}^6\frac{f(q^7)}{\{f(q)\}^7}$$

From the above reasoning we can see that the coefficient of q^{7n} on the RHS is a multiple of 7 and since $f(q^7)$ consists only of terms of the form q^{7n} it follows that the coefficient of q^{7n} in $q^2/f(q)$ is a multiple of 7. It follows that $p(7n+5)\equiv 0\pmod{7}$.

Second Proof: Let J denote any power series with integer coefficients. We can clearly see that

$$\begin{aligned} R &= 1 + 7J \\ \Rightarrow R^2 &= 1 + 7J \end{aligned}$$

and

$$\begin{aligned} Q^2 - PR &= -2q\frac{dR}{dq} = 1008J = 7J \\ \Rightarrow Q^2 &= PR + 7J = P + 7J \\ \Rightarrow Q^3 &= PQ + 7J \end{aligned}$$

Thus we can write

$$\begin{aligned} (Q^3 - R^2)^2 &= (PQ - 1 + 7J)^2 \\ &= P^2Q^2 - 2PQ + 1 + 7J \\ &= P^3 - 2PQ + R + 7J \\ &= P(P^2 - Q) - (PQ - R) + 7J \\ &= 12qP\frac{dP}{dq} - 3q\frac{dQ}{dq} + 7J \\ &= 6q\frac{dP^2}{dq} - 3q\frac{dQ}{dq} + 7J \\ &= q\frac{dJ}{dq} + 7J \end{aligned} \tag{1}$$

Again it is obvious that

$$\begin{aligned} (1-q)^7 &= 1 - q^7 + 7J \\ \Rightarrow (1-q)^{49} &= (1-q^7)^7 + 7J = 1 - q^{49} + 7J \end{aligned}$$

Replacing q by $q^2, q^3, \ldots$ and then multiplying the resulting equations we get

$$\begin{aligned} \{f(q)\}^{49} &= f(q^{49}) + 7J \\ \Rightarrow \{f(q)\}^{48} &= \frac{f(q^{49})}{f(q)} + 7J \end{aligned} \tag{2}$$

From equation (1) we get

$$1728^2q^2\{f(q)\}^{48} = q\frac{dJ}{dq} + 7J$$

and using equation (2) we can rewrite above equation as

$$\begin{aligned} 1728^2q^2\frac{f(q^{49})}{f(q)} &= q\frac{dJ}{dq} + 7J \\ \Rightarrow 1728^2\frac{q^2}{f(q)} &= q\frac{dJ}{dq}\frac{1}{f(q^{49})} + 7J \end{aligned}$$

Clearly the coefficient of q^{7n} in qdJ/dq is a multiple of 7 and therefore the coefficient of q^{7n} in RHS of the above equation is a multiple of 7. Hence the coefficient of q^{7n} in $q^2/f(q)$ is also a multiple of 7. Thus we get $p(7n+5) \equiv 0 \pmod{7}$.

Proof of $p(11n+6)\equiv 0 \pmod{11}$

The proof of this congruence is based on the properties of functions P, Q, R and is bit complicated in terms of algebraic manipulations. The idea is same as the one used for congruences modulo 5 and 7. We have to establish the identity

$$(Q^3-R^2)^5 = q\frac{dJ}{dq}+11J \tag{3}$$

where J represents a power series with integer coefficients. To that end we first need to simplify the expression $(Q^3-R^2)^5$. We have from equations $(26), (27)$ of this post

$$QR = 1+11J$$

and

$$\begin{aligned}Q^3-3R^2 &= 441Q^3+250R^2+11J\\ &= 691+65520\Phi_{0,11}(q)+11J\\ &= -2+48\Phi_{0,11}(q)+11J\\ &= -2+48\Phi_{0,1}(q)+11J\\ &= -2P+11J\end{aligned}$$

where we have used the fact that $n^{11}\equiv n \pmod{11}$ so that $\Phi_{0,11}(q)=\Phi_{0,1}(q)+11J$.

Now Ramanujan does something which is awesome. He writes that *"it can be easily deduced that"*

$$(Q^3-R^2)^5 = (Q^3-3R^2)^5-Q(Q^3-3R^2)^3-Q(Q^3-3R^2)^2+6QR+11J$$

It may have been quite easy for Ramanujan but in reality it is a tedious computation done as follows (making use of the fact that $QR=1+11J$):

$$\begin{aligned}&(Q^3-3R^2)^5-Q(Q^3-3R^2)^3-Q(Q^3-3R^2)^2+6QR\\ &\quad= (Q^3-3R^2)^5-Q^3R^2(Q^3-3R^2)^3-Q^3R^4(Q^3-3R^2)^2+6Q^6R^6+11J\\ &\quad= Q^{15}-16Q^{12}R^2+98Q^9R^4-285Q^6R^6+423Q^3R^8-243R^{10}+11J\\ &\quad= Q^{15}-5Q^{12}R^2+10Q^9R^4-10Q^6R^6+5Q^3R^8-R^{10}+11J\\ &\quad= (Q^3-R^2)^5+11J\end{aligned}$$

and therefore we can write

$$\begin{aligned}(Q^3-R^2)^5 &= (Q^3-3R^2)^5-Q(Q^3-3R^2)^3-Q(Q^3-3R^2)^2+6QR+11J\\ &= -32P^5+8P^3Q-4P^2Q+6QR+11J\\ &= P^5-3P^3Q-4P^2Q+6QR+11J\end{aligned} \tag{4}$$

Next Ramanujan probably does a hell lot of calculations to show that the expression on RHS can be written in the form qdJ/dq and to simplify these calculations Ramanujan introduces a symbol Θ as a shorthand for the operation $q\dfrac{d}{dq}$. We start with

$$\begin{aligned}\Theta P &= \frac{P^2 - Q}{12}\\ \Theta Q &= \frac{PQ - R}{3}\\ \Theta R &= \frac{PR - Q^2}{2}\end{aligned}$$

Ramanujan does not provide the details of the calculation and instead writes that from the above equations connecting P, Q, R the final result (equation (3) above) can be easily deduced. Details of the calculation were finally provided by G. H. Hardy. Following Hardy we write

$$\begin{aligned}\Theta^2 P &= \Theta(\Theta P)\\ &= \Theta\left(\frac{P^2 - Q}{12}\right)\\ &= \frac{1}{12}(\Theta(P^2) - \Theta Q)\\ &= \frac{1}{12}\left(2P\Theta P - \frac{PQ - R}{3}\right)\\ &= \frac{1}{12}\left(2P\frac{P^2 - Q}{12} - \frac{PQ - R}{3}\right)\\ &= \frac{2P^3 - 6PQ + 4R}{144}\\ \Rightarrow 144\Theta^2 P &= 2P^3 - 6PQ + 4R\\ \Rightarrow 144\Theta^3 P &= \Theta(2P^3 - 6PQ + 4R)\\ &= 6P^2\Theta P - 6P\Theta Q - 6Q\Theta P + 4\Theta R\\ &= \frac{P^2(P^2 - Q)}{2} - 2P(PQ - R) - \frac{Q(P^2 - Q)}{2} + 2(PR - Q^2)\\ &= \frac{(P^2 - Q)^2}{2} + 4PR - 2P^2Q - 2Q^2\\ \Rightarrow 288\Theta^3 P &= (P^2 - Q)^2 + 8PR - 4P^2Q - 4Q^2\\ \Rightarrow 288\Theta^4 P &= \Theta\{(P^2 - Q)^2 + 8PR - 4P^2Q - 4Q^2\}\\ &= 2(P^2 - Q)\Theta(P^2 - Q) + 8P\Theta R + 8R\Theta P - 8PQ\Theta P - 4P^2\Theta Q - 8Q\Theta Q\\ &= (4P^3 - 12PQ + 8R)\Theta P - (6P^2 + 6Q)\Theta Q + 8P\Theta R\\ &= \frac{(P^3 - 3PQ + 2R)(P^2 - Q)}{3} - 2(P^2 + Q)(PQ - R) + 4P(PR - Q^2)\end{aligned}$$

so that we finally arrive at

$$864\Theta^4 P = P^5 - 10P^3Q - 15PQ^2 + 20P^2R + 4QR$$

In a similar manner we calculate $\Theta^3 Q$ and $\Theta^2 R$. We put the results in the form of a table as follows:

$$\begin{array}{rcrcrcrcrcr} 864\Theta^4 P & = & P^5 & - & 10P^3Q & - & 15PQ^2 & + & 20P^2R & + & 4QR\\ 72\Theta^3 Q & = & & & 5P^3Q & + & 15PQ^2 & - & 15P^2R & - & 5QR\\ 24\Theta^2 R & = & & & & - & 14PQ^2 & + & 7P^2R & + & 7QR\end{array}$$

Clearly the LHS of each of these equations is of the form qdJ/dq and hence on multiplying first of these by 1, second by 8 and third by 2 and on adding the resulting equations modulo 11 we get

$$P^5 - 3P^3Q + 2P^2R = q\frac{dJ}{dq} + 11J \tag{5}$$

Next we have

$$P^2 - Q = 12q\frac{dP}{dq}$$

so that

$$6P^2R - 6QR = 72qR\frac{dP}{dq}$$

On the other hand we can see that

$$\begin{aligned}72qR\frac{dP}{dq} &= -36qR\frac{d}{dq}(-2P)\\ &= -36qR\frac{d}{dq}\{Q^3 - 3R^2 + 11J\}\\ &= 36qR\left(6R\frac{dR}{dq} - 3Q^2\frac{dQ}{dq} + 11J\right)\\ &= 72q\frac{dR^3}{dq} - 54qQR\frac{dQ^2}{dq} + 11J\\ &= 72q\frac{dR^3}{dq} - 54q\frac{dQ^2}{dq} + 11J\\ &= q\frac{d}{dq}\{72R^3 - 54Q^2\} + 11J\\ &= q\frac{dJ}{dq} + 11J\end{aligned}$$

and therefore

$$6P^2R - 6QR = q\frac{dJ}{dq} + 11J \tag{6}$$

Subtracting equation (6) from equation (5) we get

$$P^5 - 3P^3Q - 4P^2R + 6QR = q\frac{dJ}{dq} + 11J$$

and by using equation (4) we arrive at

$$\begin{aligned}(Q^3 - R^2)^5 &= q\frac{dJ}{dq} + 11J\\ \Rightarrow 1728^5q^5\{f(q)\}^{120} &= q\frac{dJ}{dq} + 11J\end{aligned} \tag{7}$$

Again it is easy to observe that

$$(1-q)^{121} = 1 - q^{121} + 11J$$

and hence by replacing q by $q^2, q^3, \ldots$ and on multiplying these equations we arrive at

$$\begin{aligned}\{f(q)\}^{121} &= f(q^{121}) + 11J\\ \Rightarrow \{f(q)\}^{120} &= \frac{f(q^{121})}{f(q)} + 11J\end{aligned} \tag{8}$$

From equations (7) and (8) we see that

$$\begin{aligned}1728^5 q^5 \frac{f(q^{121})}{f(q)} &= q\frac{dJ}{dq} + 11J\\ \Rightarrow 1728^5 \frac{q^5}{f(q)} &= q\frac{dJ}{dq}\frac{1}{f(q^{121})} + 11J\end{aligned}$$

Clearly the coefficient of q^{11n} on the RHS is a multiple of 11 and therefore the same should be the case for the corresponding coefficients on LHS. Therefore we get $p(11n+6) \equiv 0 \pmod{11}$.

From the proof given above it is quite obvious that Ramanujan was an expert in symbolic manipulation and unusually smart in finding algebraic relationships between various expressions. To date this this feat of Ramanujan is unparalleled and with the advent of computer algebra packages like MAPLE and MATHEMATICA it is almost impossible to find such talent in current times.

The technique used in the above proof is quite general and Ramanujan used it to establish partition congruences modulo powers of $5, 7, 11$. We will illustrate this point by establishing another partition congruence of Ramanujan namely

$p(25n-1) \equiv 0 \pmod{25}$

We first notice that

$$\begin{aligned}Q^3 - R^2 &= 2(Q^2 - PR) - (Q - P^2) + Q(Q-1)^2 - (R-P)^2\\ &= 2(Q^2 - PR) - (Q - P^2) + 25J\end{aligned}$$

From this post we know that

$$\begin{aligned}Q - P^2 &= 288\Phi_{1,2}(q) = 288\sum_{n=1}^{\infty} n\sigma_1(n)q^n\\ Q^2 - PR &= 1008\Phi_{1,6}(q) = 1008\sum_{n=1}^{\infty} n\sigma_5(n)q^n\end{aligned}$$

and therefore

$$\begin{aligned}Q^3 - R^2 &= \sum_{n=1}^{\infty}\{2016n\sigma_5(n) - 288n\sigma_1(n)\}q^n + 25J\\ \Rightarrow 1728q\{f(q)\}^{24} &= \sum_{n=1}^{\infty}\{2016n\sigma_5(n) - 288n\sigma_1(n)\}q^n + 25J\\ \Rightarrow 3q\{f(q)\}^{24} &= \sum_{n=1}^{\infty}\{16n\sigma_5(n) - 13n\sigma_1(n)\}q^n + 25J\\ \Rightarrow 6q\{f(q)\}^{24} &= \sum_{n=1}^{\infty}\{32n\sigma_5(n) - 26n\sigma_1(n)\}q^n + 25J\\ \Rightarrow 6q\{f(q)\}^{24} &= \sum_{n=1}^{\infty}\{7n\sigma_5(n) - n\sigma_1(n)\}q^n + 25J \qquad (9)\end{aligned}$$

Next we can see that

$$(1-q)^5 = 1 - q^5 + 5J$$

so that

$$(1-q)^{25} = \{(1-q^5) + 5J\}^5 = (1-q^5)^5 + 25J$$

and therefore

$$\begin{aligned}\{f(q)\}^{25} &= \{f(q^5)\}^5 + 25J\\ \Rightarrow \{f(q)\}^{24} &= \frac{\{f(q^5)\}^5}{f(q)} + 25J\end{aligned}$$

Using equation (9) we now arrive at

$$\begin{aligned}6q\frac{\{f(q^5)\}^5}{f(q)} &= \sum_{n=1}^{\infty}\{7n\sigma_5(n) - n\sigma_1(n)\}q^n + 25J\\ \Rightarrow 6\{f(q^5)\}^5\sum_{n=1}^{\infty}p(n-1)q^n &= \sum_{n=1}^{\infty}\{7n\sigma_5(n) - n\sigma_1(n)\}q^n + 25J\end{aligned}$$

If we extract terms containing q^{5n} we can see that

$$6\{f(q^5)\}^5\sum_{n=1}^{\infty}p(5n-1)q^{5n} = \sum_{n=1}^{\infty}\{35n\sigma_5(5n) - 5n\sigma_1(5n)\}q^{5n} + 25J(q^5)$$

where $J(q^5)$ is a power series in q^5 with integer coefficients. Replacing q^5 by q we get

$$6\{f(q)\}^5\sum_{n=1}^{\infty}p(5n-1)q^n = \sum_{n=1}^{\infty}\{10n\sigma_5(5n) - 5n\sigma_1(5n)\}q^n + 25J \tag{10}$$

Since we have $n^5 \equiv n \pmod{5}$ therefore

$$\sigma_5(n) - \sigma_1(n) \equiv 0 \pmod{5} \tag{11}$$

Replacing n by $5n$ we can see that

$$\sigma_5(5n) - \sigma_1(5n) \equiv \sigma_5(n) - \sigma_1(n) \pmod{5}$$

and multiplying by $5n$ throughout we get

$$5n\sigma_5(5n) - 5n\sigma_1(5n) \equiv 5n\sigma_5(n) - 5n\sigma_1(n) \pmod{25} \tag{12}$$

Also since σ_5 is multiplicative therefore $\sigma_5(5n) = \sigma_5(5)\sigma_5(n)$ if n is not divisible by 5 and therefore $\sigma_5(5n) \equiv \sigma_5(n) \pmod{5}$ if $5 \nmid n$. Thus we can see that $n\sigma_5(5n) \equiv n\sigma_5(n) \pmod{5}$ for all values of n. Using this result with equation (12) we see that

$$10n\sigma_5(5n) - 5n\sigma_1(5n) \equiv 10n\sigma_5(n) - 5n\sigma_1(n) \pmod{25}$$

Using equation (11) we get

$$10n\sigma_5(5n) - 5n\sigma_1(5n) \equiv 5n\sigma_1(n) \pmod{25}$$

Hence the equation (10) can be rewritten as

$$6\{f(q)\}^{5}\sum_{n=1}^{\infty}p(5n-1)q^{n}=5\sum_{n=1}^{\infty}n\sigma_{1}(n)q^{n}+25J$$

Again since $(1-q)^{5}=(1-q^{5})+5J$ it follows that $\{f(q)\}^{5}=f(q^{5})+5J$ and therefore we get

$$\{f(q^{5})+5f(q^{5})+30J\}\sum_{n=1}^{\infty}p(5n-1)q^{n}=5\sum_{n=1}^{\infty}n\sigma_{1}(n)q^{n}+25J$$

and clearly $p(5n-1)$ is a multiple of 5 therefore we get

$$f(q^{5})\sum_{n=1}^{\infty}p(5n-1)q^{n}=5\sum_{n=1}^{\infty}n\sigma_{1}(n)q^{n}+25J \tag{13}$$

The coefficient of q^{5n} on the RHS of the above equation is a multiple of 25 and therefore the coefficient of q^{5n} on the LHS of (13) should also be a multiple of 25. It follows that $p(25n-1)\equiv 0\pmod{25}$. Another marvelous proof (also by Ramanujan) of this congruence is presented in a later post.

Note: The content of this post and the previous one is from Ramanujan's papers titled *Some Properties of Partitions* and *Congruence Properties of Partitions*. Dr. Ken Ono and Dr. Bruce C. Berndt have rewritten the paper of Ramanujan alongwith with missing proofs and commentary and the same is available here.

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