Section 1.3: Equal Temperament

Both of the tuning systems we've considered thus far – Pythagorean and just intonation – use frequency ratios to determine which tones to include within each octave. Pythagorean is based entirely on preserving all octaves and (almost) all perfect fifths; just intonation also preserves all octaves, but allows certain fifths to become imperfect, or "broken," in order to preserve some other desirable intervals (such as thirds and sixths). The next system we'll consider, called "equal temperament," is based on an entirely different consideration.

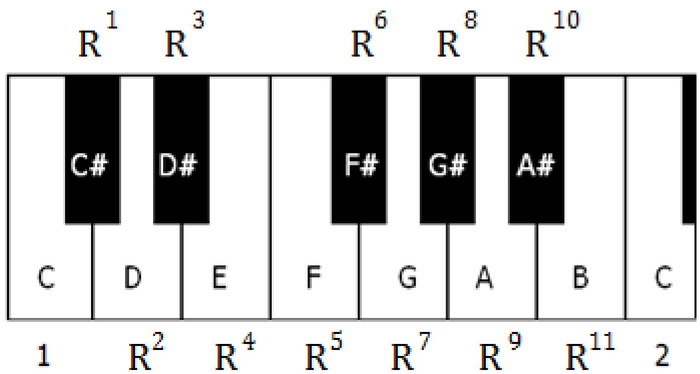
One drawback of both systems considered thus far is the problem of inconsistent semitones – that is, each individual step (actually called a "half step" in musical terminology) does not have the same frequency ratio as each of the others. For example, in a Pythagorean tuning system based on "C," the first semitone, C-C#, has a frequency ratio of 256/243 (or about 1.0535), while the second semitone, C#-D, has frequency ratio of 2187/2048 (1.0679). This inconsistency – which persists throughout the scale – makes it impossible to "transpose" a melody or chord without fundamentally changing its sound. Just intonation has this drawback as well (as seen in the homework).

The desire for a tuning system that allows for transposition, while still almost preserving desirable just intonation intervals such as thirds, fourths, fifths, and sixths, is what leads to the development of "equal temperament." This new tuning system is designed based on the following criteria:

- Preserve all octaves – that is, pairs of notes separated by an octave will always have a frequency ratio of <u>exactly</u> 2/1
- Consistent semitones –pairs of notes separated by 1 semitone will always have the same frequency ratio – no exceptions!

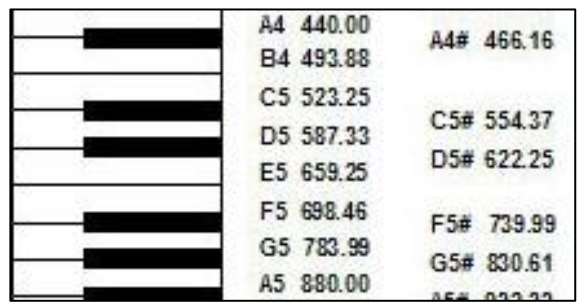
The trick to constructing 12-tone equal temperament (hereafter referred to as "12-TET") is to figure out what semitone frequency ratio we need to use in order to preserve both consistent semitones and octaves. If we have twelve semitones to the octave, then raising a note by semitones twelve times must have the cumulative effect of raising it by an octave. Mathematically: if R is the frequency ratio of a semitone, then R multiplied by itself 12 times – that is, R^{12} - must be equal to 2. The only frequency ratio with this desired property is the twelfth root of 2: $R = \sqrt[12]{2}$, or about 1.0595. Under 12-TET, **every** semitone is tuned according to this frequency ratio.

For our other tuning systems, we presented a keyboard diagram summarizing one octave of frequencies in the system. The corresponding diagram for 12-TET is shown below, with "R" inserted in place of $\sqrt[12]{2}$:



For example, consider the piano frequencies diagram that appeared on the first page of Section 1.0. This diagram, which shows the actual tuning used on most modern keyboards, is based on the 12-TET system described above, with A4 (the fourth A on the keyboard) tuned to exactly 440 Hz. Below we'll see how the frequencies of the notes in the octave from A4:440 to A5:880 are obtained:

A4: 440 Hz
A#4: $440 \times \sqrt[12]{2} \approx 466.16$ Hz
B4: $440 \times \left(\sqrt[12]{2}\right)^2 \approx 493.88$ Hz
C4: $440 \times \left(\sqrt[12]{2}\right)^3 \approx 523.25$ Hz
…and so on, as shown in the diagram to the right.



Note that the next higher A has frequency
A5: $440 \times \left(\sqrt[12]{2}\right)^2 = 440 \times 2 = 880$ Hz,
as required.

Now that all semitones have the same frequency ratio, we can easily find frequency ratios for other intervals as well based on their widths in semitones. For example, consider the "perfect fifth" interval, whose width is traditionally seven semitones. (Recall that the "ideal" frequency ratio of a perfect fifth is 3/2, or 1.5.) We'll consider the A-E perfect fifth as a working example. If we are given the frequency of the A, then we must raise by a semitone 7 times to find the frequency of the next higher E. This requires multiplying the frequency of the A by $\sqrt[12]{2}$ seven times; or, equivalently, it requires multiplying by $\left(\sqrt[12]{2}\right)^7$, which (use your calculator!) is approximately 1.4983. This is the 12-TET frequency ratio for all seven-semitone intervals. (Note that it is not exactly 1.5… but, it *is* _very_ close!)

As another example, consider the major third, which is a four-semitone interval. By the same reasoning used in the preceding paragraph, the 12-TET frequency ratio of a major third would be $\left(\sqrt[12]{2}\right)^4$, or approximately 1.2599. (Again, note that this is close to the "ideal" frequency ratio of 5/4, or 1.25.) It's instructive to find this frequency ratio in yet another way: note that a major third (four semitones) is exactly 1/3 of an octave (12 semitones). This means the frequency ratio of a major third should be a number, we'll call it R for now, such that multiplying by T three times gives us a result of 2. In other "words," we should have $R^3 = 2$, which implies $R = \sqrt[3]{2}$. If you enter this into your calculator, you'll find that this result, approximately 1.2599, is consistent with the $\left(\sqrt[12]{2}\right)^4$ we obtained earlier, as it should be!

The point of the preceding example is to justify a change in notation, which we will use from here on out. In general, the number $\sqrt[2]{2}$ may be written in the exponential form $2^{\frac{1}{12}}$, and $\left(\sqrt[12]{2}\right)^n$ (where n is any whole number) may be written as $2^{n/12}$. More generally, the nth root of any number can be written using the exponent 1/n – that is,

$$\boldsymbol{\sqrt[n]{a} = a^{1/n}}$$

where "a" can be any positive number.

Further, using the "power-of-a-power rule" for exponents (see section 1.3.2 if this is unfamiliar), it follows that

$$\boldsymbol{\left(\sqrt[n]{a}\right)^m = 2^{m/n}}.$$

In particular, in the 12-TET tuning system, *every* m-semitone interval has the frequency ratio $2^{m/12}$.

So, for example, refer back a few paragraphs to the discussion of a major third (four semitones) under 12-TET. The frequency ratio of this interval, $\left(\sqrt[12]{2}\right)^4$, can be rewritten as $2^{4/12}$. Also note that the fraction 4/12 reduces to 1/3, which means our frequency ratio can also be rewritten as $2^{1/3}$; this corresponds to the observation that the frequency ratio must also be equal to $\sqrt[3]{2}$.

For further review of properties of exponents (and logarithms), see Section 1.3.2.)

<u>Practice Exercises</u> (answers are on the next page):

1. Suppose a standard 12-tone keyboard is tuned using 12-TET with an A tuned to 450 Hz. Find the frequency of the next higher C, the next higher E, and the next lower D.

2. Suppose a pentatonic scale is tuned using 5-TET, with base note A:440 Hz. Find the frequencies of the other notes in the pentatonic scale (up to A:880 Hz).

OK, I've got this one tuned to 450… now what? I'm just a kitten, help me out here…

Practice Exercise Answers

1. Keep in mind that 12-TET frequency ratios depend entirely on the number of semitones between the upper and lower notes:
   - Next higher C: first notice that A-C is a three-semitone interval. Therefore, the corresponding frequency ratio is $2^{3/12}$ (which is about 1.189). So, to raise from A:450 to the next higher C, we'd multiply by this frequency ratio:
   $$\mathbf{450 \times 2^{3/12} \approx 535.14\ Hz}$$

     Note: If you round off $2^{3/12}$ to 1.189 before multiplying, you end up with 535.05 – this is not a huge error (only about 0.1 Hz), but it's best to avoid rounding errors whenever possible. It is strongly recommended to let the calculator do its job, by using exact quantities at each step of a calculation rather than rounding off at intermediate steps. (

   - Next higher E: This is very similar to the previous part of the problem; the only difference is that, since C-E is a seven semitone interval, we'll multiply by the frequency ratio $2^{7/12}$:
   $$450 \times 2^{7/12} \approx 674.24\ Hz$$

   - Next lower D: The number of semitones between an A and the next *lower* D is seven, so our frequency ratio is again $2^{7/12}$. However, since we're *lowering* the pitch this time, we will *divide* by this frequency ratio to tune the next *lower* D:
   $$450 \div 2^{7/12} \approx 300.34\ Hz$$

     Alternate solution: Another way to approach this would be to start by dropping down to the next-lower A, whose frequency is $450 \div 2 = 225\ Hz$. Then, we could count from that A up to the next D, finding that A-D is a five semitone interval. Therefore, we'd raise A:225 by the frequency ratio $2^{5/12}$:
   $$225 \times 2^{\frac{5}{12}} \approx 300.34\ Hz$$

     Notice that dividing by $2^{\frac{7}{12}}$ has the same effect as dividing by 2 and then multiplying by $2^{\frac{5}{12}}$. This is a consequence of the division rule for exponents (covered in more detail in section 1.3.2).

2. Under 5-TET, the frequency ratio of one "semitone" would be $2^{1/5}$ rather than $2^{\frac{1}{12}}$. So, to find the frequency of each note in the 5-TET pentatonic scale, we'd start from A:440 and repeatedly multiply by $2^{1/5}$ (which, incidentally, is about 1.1487).
   - 440 Hz
   - $440 \times 2^{1/5} \approx 505.43\ Hz$
   - $\left(440 \times 2^{1/5}\right) \times 2^{1/5} \approx 580.58\ Hz$ (or, equivalently: $440 \times 2^{2/5} \approx 580.58$)
   - $440 \times 2^{3/5} \approx 666.92\ Hz$
   - $440 \times 2^{4/5} \approx 766.08\ Hz$
   - $440 \times 2 = 880\ Hz$

   Side note: compare these results to the 12-TET piano tuning table from earlier in this section. What 12-TET frequencies are the closest matches to these pentatonic frequencies? For example, 505.43 Hz is between B:493.88 and C:523.25, but slightly closer to B. How about the others