

## Limits at Infinity and Asymptotes

(based on a lab of the same name written by Drs. Bergner and Spickler)

Recall that a horizontal asymptote indicates a limit at infinity or negative infinity.

*Mathematica* can calculate limits at infinity (or negative infinity) using the Limit command.

For example, try these commands:

```
f[x_] := (2 x^2 - 3 x + 4) / (x^2 - 5 x + 9)
```

```
f[x]
```

$$\frac{4 - 3x + 2x^2}{9 - 5x + x^2}$$

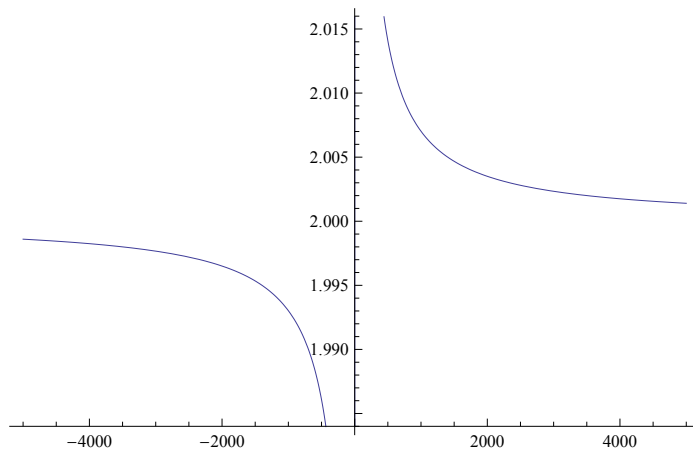
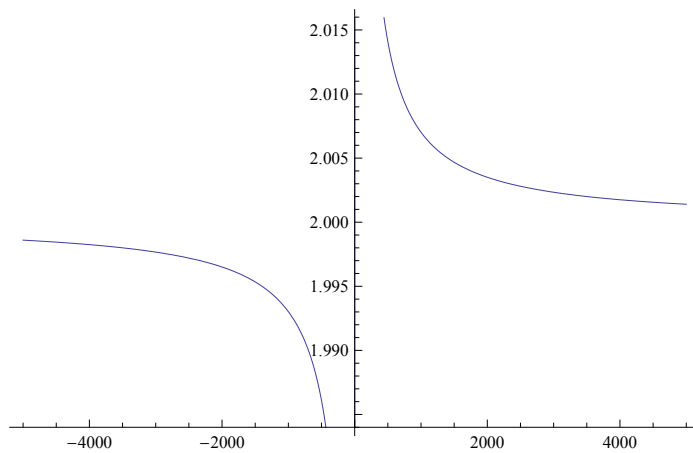
```
Limit[f[x], x -> Infinity]
```

```
2
```

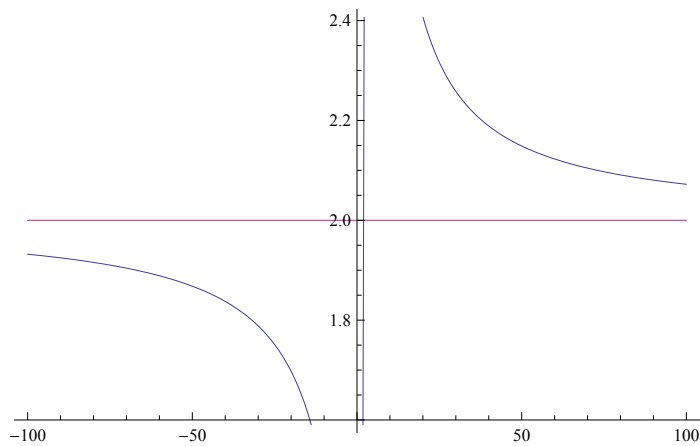
```
Limit[f[x], x -> -Infinity]
```

```
2
```

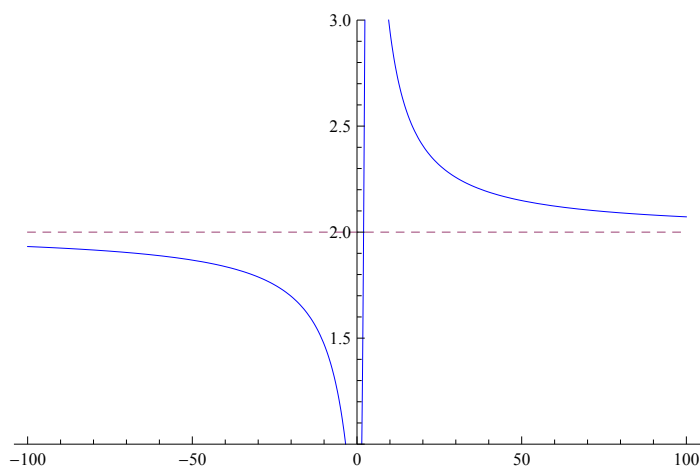
```
Plot[f[x], {x, -5000, 5000}]
```



```
Plot[{f[x], 2}, {x, -100, 100}]
```



```
Plot[{f[x], 2}, {x, -100, 100}, PlotRange -> {1, 3}, PlotStyle -> {Blue, Dashed}]
```



Notice the differences among the above plots. In order to clearly demonstrate the function's behavior as  $x$  goes to infinity in either direction, it may be necessary to manipulate the `PlotRange` option. Also, `PlotStyle` allows us to clearly distinguish between the graph of the function and the asymptote.

Another example is given below. In this case, note that the limits at positive infinity and negative infinity are different. (Recall that the "arctangent" is the inverse tangent function; specifically, it's the inverse of the tangent function whose domain is restricted to  $(-\pi/2, \pi/2)$ ).

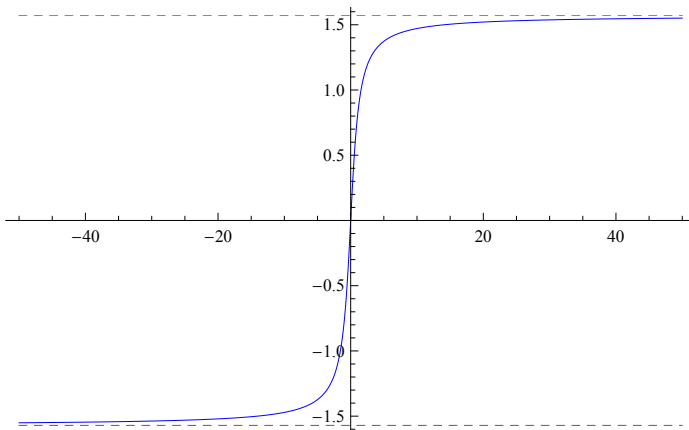
```
Limit[ArcTan[x], x -> Infinity]
```

$$\frac{\pi}{2}$$

```
Limit[ArcTan[x], x -> -Infinity]
```

$$-\frac{\pi}{2}$$

```
Plot[{ArcTan[x], -Pi / 2, Pi / 2}, {x, -50, 50}, PlotStyle -> {Blue, Dashed, Dashed}]
```



More generally, any line (or other curve) that a given function approaches as  $x$  approaches either infinity can be regarded as an asymptote. Consider the following example:

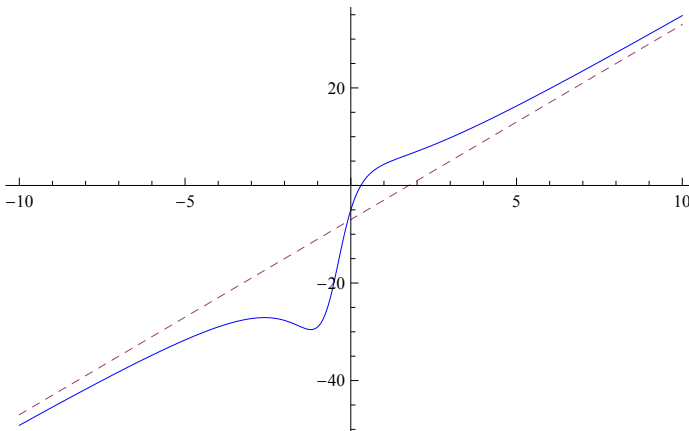
```
Clear[f]
```

(Note : the above command "clears" the definition of "f". It's a good habit to do this before redefining any symbol -- e.g., f, x, etc. -- within the same notebook.)

```
f[x_] := (4 x^3 - 3 x^2 + 17 x - 5) / (x^2 + x + 1)
```

```
q[x_] := 4 x - 7
```

```
Plot[{f[x], q[x]}, {x, -10, 10}, PlotStyle -> {Blue, Dashed}]
```



In this example, we would call  $q$  an "asymptotic function" - more precisely, we would say that  $f$  is *asymptotic to*  $q$ , which means that the *difference* between these two functions approaches zero as  $x$  approaches infinity (or negative infinity).

For rational functions, finding an asymptotic function is simply a matter of polynomial long division : for example,  $4x - 7$  is the quotient when  $4x^3 - 3x^2 + 17x - 5$  is divided by  $x^2 + x + 1$ .

We can use *Mathematica* to carry out polynomial long division for us as follows:

```
PolynomialQuotient[4 x^3 - 3 x^2 + 17 x - 5, x^2 + x + 1, x]
- 7 + 4 x
```

This is the asymptote in the above graph (we called it  $q[x]$ ).

As an aside - although we won't need it for this lab assignment, there's also a command which tells us the remainder obtained from the long division of polynomials:

```
PolynomialRemainder[4 x^3 - 3 x^2 + 17 x - 5, x^2 + x + 1, x]
2 + 20 x
```

Thus,  $f[x] = 4x - 7 + (20x + 2)/(x^2 + x + 1)$ .

When written in this form, we see that  $f[x]$  is equal to  $4x-7$  plus a rational function which approaches zero as  $x$  approaches infinity in either direction.

$$\text{Specifically, } f(x) = \frac{4x^3 - 3x^2 + 17x - 5}{x^2 + x + 1} = 4x - 7 + \frac{20x + 2}{x^2 + x + 1}.$$

Let's look at one more example:

```
Clear[f]
```

```
f[x_] := (x^4 - 10 x^3 + 3 x^2 - 7 x + 9) / (2 x^2 + 9 x - 5)
```

```
f[x]
```

$$\frac{9 - 7x + 3x^2 - 10x^3 + x^4}{-5 + 9x + 2x^2}$$

```
PolynomialQuotient[x^4 - 10 x^3 + 3 x^2 - 7 x + 9, 2 x^2 + 9 x - 5, x]
```

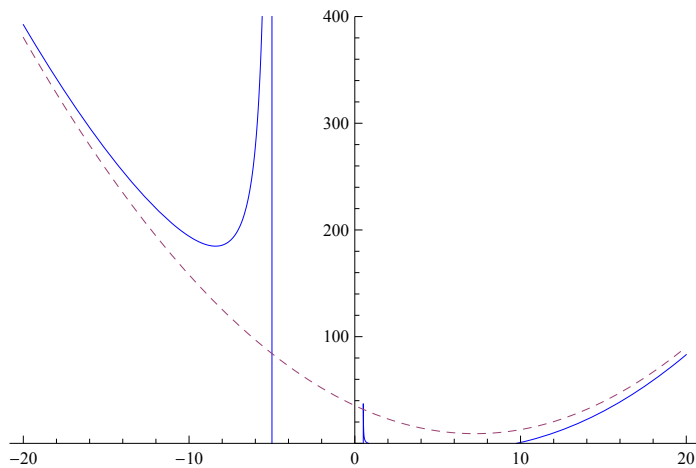
$$\frac{283}{8} - \frac{29x}{4} + \frac{x^2}{2}$$

$$Q[x_] := \frac{283}{8} - \frac{29x}{4} + \frac{x^2}{2}$$

```
Q[x]
```

$$\frac{283}{8} - \frac{29x}{4} + \frac{x^2}{2}$$

```
Plot[{f[x], Q[x]}, {x, -20, 20}, PlotRange -> {0, 400}, PlotStyle -> {Blue, Dashed}]
```

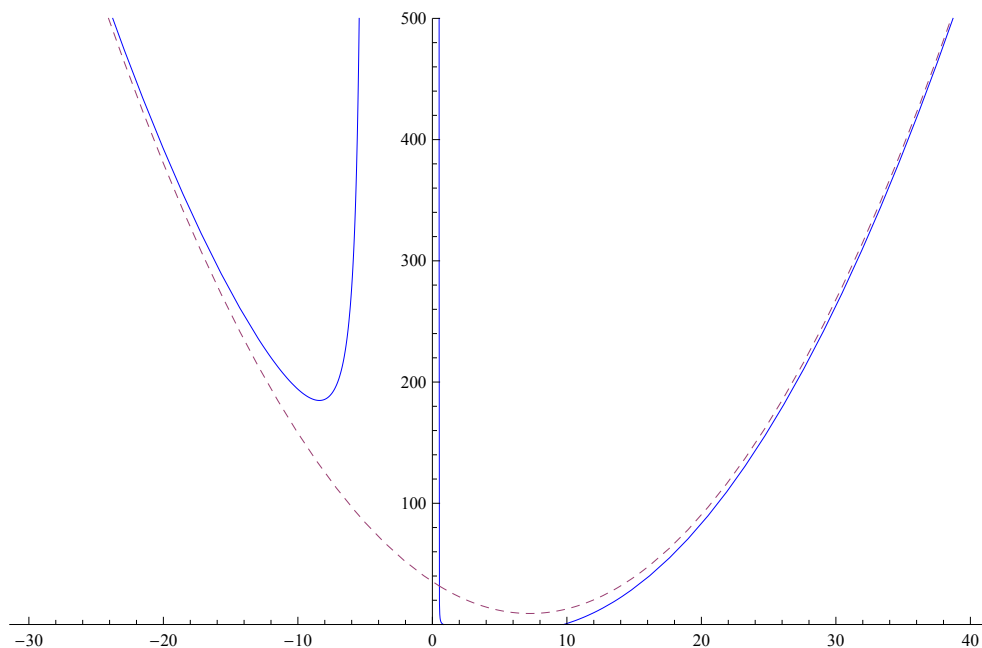


Note : In order to get rid of the vertical slashes (which occur where  $f[x]$  has infinite discontinuities), we'd need to find the locations of these discontinuities, then use the "Exclusions" option to tell Mathematica not to try to graph  $f[x]$  at these numbers. While it's not strictly necessary for the purposes of this lab, it's still a good idea (for aesthetic reasons, if nothing else):

```
In[10]:= Solve[-5 + 9 x + 2 x^2 == 0, x]
```

```
Out[10]= {{x -> -5}, {x -> 1/2}}
```

```
Plot[{f[x], Q[x]}, {x, -30, 40}, PlotStyle -> {Blue, Dashed},  
PlotRange -> {0, 500}, Exclusions -> {-5, 1/2}]
```



Note : The above x-interval and PlotRange took some experimentation; part of your job on this lab assignment is to find a viewing window that does a good job of demonstrating to the viewer the asymp-

otic behavior of the functions being graphed. This will take some experimentation (i.e., trial-and-error) on your part. Don't just print out the default graph; in most cases, that is not adequate!

## Lab Exercises:

For each function, find an asymptotic function by the method illustrated in the preceding examples. Graph the function along with its asymptotic function, and choose a viewing window (by experimenting with the x-interval and the PlotRange option) in such a way that the graph *clearly* displays the asymptotic relationship between the two functions.

**Make sure that all the results and graphs show up on your printout. Print out the completed notebook, and turn it in on Monday, September 28.**

$$1. f[x] = \frac{x^2 - 25x + 2}{6x^2 + x - 2}$$

$$2. g[x] = \frac{x^4 + 5x^2 - 7x + 2}{2x^2 + x - 2}$$

$$3. h[x] = \frac{4x^3 + 3x^2 - 4x + 26}{2x^2 + x - 2}$$

$$4. j[x] = \frac{x^5 - 4x^3 + 25x + 2}{x^2 + x - 2}$$

$$5. k[x] = \frac{x^5 - 10000}{x^4 + 100x^3 + 10000}$$