Section 1.3.3: Rational Approximation

Rational Approximation is a process by which a real number is "approximated" by a rational number. (A "rational number" is simply a fraction, a/b, where a and b are both whole numbers.) It is typically used to find a relatively "simple" fraction that can be used to estimate another number, such as an irrational number or a fraction with a much larger denominator.

We've seen examples of this idea already in the transition from Pythagorean tuning to just intonation. For instance, the frequency ratio of a major third is 81/64 (or 1.265625) under Pythagorean tuning; under just intonation, this fraction is approximated with the much "simpler" fraction 5/4 (or 1.25).

So, we've already seen that one music-related motivation for learning about rational approximation is finding "nice," just intonation frequency ratios that approximate results of other tuning systems. (For example, consider the problem of finding fractions that simulate the ratios obtained in a 16-TET tuning system.) Another motivation, discussed below, is the attempt to find equal temperament systems which are in some sense "better" than the standard 12-TET.

EXAMPLE: Finding a "better" equal temperament system than 12-TET

One reason for the appeal of 12-TET is that $2^{7/12}$, the frequency ratio of a seven-semitone interval, is a *very* close approximation to 1.5, the frequency ratio of a perfect fifth. In particular, $2^{7/12} \approx 1.4983$. This is why, on a standard (equally tempered) piano, a seven semitone interval sounds *almost exactly* like a perfect fifth. There is a small difference, but to most people it's indistinguishable.

One way to find an arguably "better" equal temperament system than 12-TET would be to find some other fraction, a/b, such that $2^{a/b}$ is even closer to 1.5 than $2^{7/12}$. If we could find such a fraction, it would follow that in an equal temperament system with b tones to the octave (rather than 12), an interval of a semitones would be a better approximation of a perfect fifth than the one we get from the 12-TET scale. In that sense, we could argue that the b-TET system is "better than" the 12-TET system.

To find this better approximation to the perfect fifth, we will need to look at fractions with larger denominators. This will lead us to equally tempered tuning systems with more than 12 tones per octave.

Here's a systematic approach: what we're really trying to do is solve the *equation* $2^x = 1.5$. (We've been using $x = 7/12$ as our "solution," though we know it's only a pretty close approximation.) As we've seen, equations of this type – with the variable in the exponent – can be solved by using logarithms:

$$\begin{aligned} 2^x &= 1.5 \\ \log(2^x) &= \log(1.5) \\ x \cdot \log(2) &= \log(1.5) \\ x &= \frac{\log(1.5)}{\log(2)} \end{aligned}$$

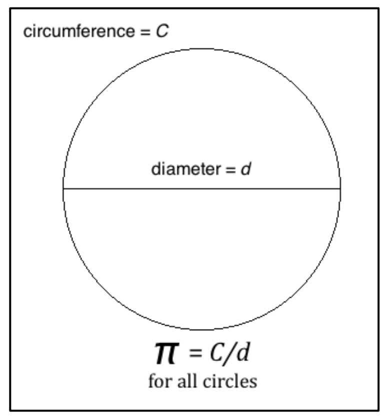
So, the *exact* solution we're looking for is $\log(1.5)/\log(2)$, which is about 0.5849625. By way of comparison, 7/12 (our current "best" rational* approximation for x in the above equation) is about

0.583333. So, 7/12 is pretty close to the exact value we're looking for. To see exactly how close, one simply way to compare is to look at the difference between these two numbers; that is, subtract one from the other: 0.5849625-0.583333 = 0.0016292, which is roughly 0.0016, or 16 ten-thousandths. That's pretty close – but, we can do better!

So, we can re-state the question: when we say we're looking for a fraction, a/b, such that $2^{a/b}$ is as close as possible to 1.5, we're really looking for a fraction, a/b, which is as close as possible to log(1.5)/log(2). One way to approach this problem would be to use a "guess-and-check" approach to try to find fractions that would serve as better approximations than 7/12. Below, we will introduce a more methodical – and very powerful – method of finding very good rational approximations to real numbers.

One way to estimate a decimal with a fraction is to find its reciprocal, and round that off to a whole number. A good way to introduce this idea is by finding rational approximations for the well-known mathematical constant pi (π). (We'll come back to log(1.5)/log(2) later!)

You may recall from high school geometry that π is the ratio between the circumference and the diameter of all circles, and its value is approximately 3.14159265…. It is actually an irrational number, which means it has an infinite, non-repeating decimal. (The number from the preceding example, log(1.5) / log(2), is also irrational.) For numbers like this – that is, numbers we need to work with, but which can't be written in an exact form – it is often helpful to find rational approximations.



EXAMPLE: Find rational approximations for π

For π, we start with the decimal, 3.14159265…, which can be found to several decimal places automatically on any scientific calculator. We can think of this as being approximately 3 plus some fraction, whose decimal value is about .1415… To find a good fraction to use, we need a trick, which is this: for decimals between 0 and 1, a good way to find an approximating fraction is to look at the reciprocal of that fraction.

We illustrate with the current example: on your calculator, pull up the decimal for π, then subtract 3. What's left is the fractional part of the number, which is about 0.1415265… Now, find the reciprocal of this. (The reciprocal key on your calculator is probably labeled either $1/x$ or x^{-1}.) This gives you a result of 7.0625…, which is very close to 7. That means the original decimal (before you took the reciprocal) was very close to the fraction 1/7. (You can check this: $\frac{1}{7} \approx 0.142857$, which is not too far off from 0.1415265.) So, we can use of $3\frac{1}{7}$, or $\frac{22}{7}$, as a "rational approximation" for π.

In fact, $\pi \approx \frac{22}{7}$ is often encouraged in computations; it's considered to be a very good approximation because it is very accurate (off by just over 0.001, making it an even better estimate than the more familiar $\pi \approx 3.14$) despite having a very small denominator. More generally, there will be a tradeoff between accuracy and simplicity – better approximations usually require much larger denominators, which can eventually render fractions to be more trouble than they're worth.

THE METHOD OF “CONTINUED FRACTIONS”

First, enter the number you want to approximate into your calculator, and get a decimal approximation. Then, carry out the following sequence of steps, as many times as you like. The more repetitions you carry out, the more precise your eventual result will be. Note that as you go through the following sequence of steps, you will be generating a LIST of whole numbers (see Step #1).

1. Write down the “whole number part” (that is, the whole number to the left of the decimal point) of the number currently on your calculator.
   *** At this point, <u>either</u> end the process, <u>or</u> go on to Step #2 ***
2. On your calculator, subtract the “whole number part” from the current number.
3. Hit the “reciprocal” button on your calculator.
   Then, REPEAT these steps, starting from Step #1.

<u>EXAMPLE</u>: Enter log(1.5)/log(2) into your calculator: **0.58496250072…**
Now, start from step 1 above…

1. Write down 0. (This will be the first “whole number part” in your list.)
2. Subtract zero. (In other words, don’t do anything.) You should still have **0.58496250072…**
3. Hit the “reciprocal” button. Result: **1.70951129135….** Now, <u>repeat</u> from Step #1…

The whole number part of 1.70951129135… is 1, so write that down.
(Note: Your current list of “whole number parts” is now [0, 1].)

1. Subtract 1. Your result should be 0.7095112915….
   (Note: the point of subtracting the “whole number part” of the current number is to give you a result between 0 and 1. If your display is ever negative or greater than 1 after this step, then you did something wrong!)
2. Hit the “reciprocal” button. Result: 1.40942083965…. Now, repeat from step 1…

1. The whole number part of **1.40942083965…** is 1, so write it down. Your list of “whole number parts” is now [0,1,1].
2. Subtract 1. Your result should be **0.40942083965….**
3. Hit the reciprocal key. Your result should be **2.44247459618….**

1. The whole number part of **2.44247459618…** is 2. Your list is now [0, 1, 1, 2].
2. Subtract 2. Result**: 0.44247459618…**
3. Reciprocal. **Result 2.26001675267…**

[NOTE: We can continue this process as many – or, as few – times as we wish. The more whole numbers we accumulate in our list, the closer our approximation will be when we’re done.]

1. The whole number part of **2.26001675267…** is 2. Current list: [0, 1, 1, 2, 2].
…we’re going to stop here for the time being. (We can always resume this process later.)

So, we have the "whole number part" list of [0, 1, 1, 2, 2]. Now, the trick is to "reverse" this list to get a fractional approximation of the original number, log(1.5) / log(2) . To do this, we work from *right to left* along our list of whole numbers, just doing the *opposite* of what we did in steps 1, 2 and 3 above.

Our current whole number list is [0, 1, 1, 2, **<u>2</u>**] . To get our rational approximation, we will start from the right of this list – that is, with the 2. Carry out the following sequence of steps as many times as necessary; this "reverses" the process carried out previously.

IMPORTANT NOTE: This time, we're NOT using a calculator! We want a <u>fraction</u>, not a decimal, so we'll need to use pencil and paper!

A. Add the next number (moving right to left) in your list of whole numbers. (Note: This "reverses" step 2 from the previous list of steps.) At the same time, cross out that whole number from your list, so that you don't accidentally use it twice. (This is an easier mistake to make than you might think!)
B. If you just added the left-most whole number in your list, then you're done!
If not, then go on to part C.
C. Find the reciprocal of your current fraction. (Note: This "reverses" step 3 in the previous list of steps.)

We'll apply the above steps to our list, [0, 1, 1, 2, **<u>2</u>**]

A. Start from the right-most 2.
(B. We're not done, so continue…)
C. the reciprocal of 2 is ½.

The next number, going right to left, is 2. After carrying out step "B", we're currently at the underlined number in the list: [0, 1, 1, **<u>2</u>**, ~~2~~]. (Note that you *must* keep track of your current location in the list for this method to work!)

A. Add 2 to the result from the previous step: $2 + \frac{1}{2} = \frac{2}{1} + \frac{1}{2} = \frac{4}{2} + \frac{1}{2} = \frac{5}{2}$.
Reminder: do NOT enter these numbers on your calculator! We're looking for a *fraction*, not a decimal. For this method to work, you'll need to manipulate fractions on paper, with your pencil! This might take a little practice, depending on how recently you last had to work with fractions…
(B. We're not done yet, so continue…)
C. Find the reciprocal: the reciprocal of 5/2 is 2/5.

Move to the left – now we've arrived at a 1. (We're currently here: [0, 1, **<u>1</u>**, ~~2, 2~~].)

A. Add this number to the current fraction: $1 + \frac{2}{5} = \frac{5}{5} + \frac{2}{5} = \frac{7}{5}$.
(B. We're not done yet, so continue…)
C. The reciprocal of7/5 is 5/7.

Move to the left; now we're here: [0, **<u>1</u>**, ~~1, 2, 2~~].

A. Add 1: $1 + \frac{5}{7} = \frac{7}{7} + \frac{5}{7} = \frac{12}{7}$
(B. Not quite done – there's one more whole number to go! So, continue one more time…)
C. The reciprocal of 12/7 is 7/12. (Look familiar?)

Move to the left: this takes us to the 0 all the way at the beginning of the list: [**0**, ~~1, 1, 2, 2~~].

A. $0+\frac{7}{12}\ =\frac{7}{12}$.

B. We are done – our answer is 7/12!

So, going through the algorithm on the preceding page five times gives us the rational approximation 7/12. We already knew about this particular estimate for log(1.5) / log(2) , but it's good to see that this "continued fractions" process is doing something like what we expected. Now, let's carry it a little bit further, and see what other rational approximations for log(1.5)/log(2) we can come up with. On the next page, we'll pick up from where we left off just before the end of page 2….

(… look back at the end of page 3; we're picking up from where we stopped before…)

1. The whole number part of 2. 26001675267 is 2. Current list: [0, 1, 1, 2, 2].

This time, let's continue on from here…

2. Subtract 2. Result: 0. 26001675267

3. Reciprocal. Result: 3.84590604155

1. The whole number part of 3.84590604155 is 3. Current list: [0, 1, 1, 2, 2, 3]

Stop here for now. (We'll do more iterations of this process a little later.

Now, let's "reverse" the list [0, 1, 1, 2, 2, 3]:

Start with 3. (Remember, keep track of the current location in the list: right now it's [0, 1, 1, 2, 2, **3**].)

A. Add 3. (B. we're not done yet, so continue)

C. Reciprocal: 1/3

Move left; the next whole number is a 2. (Now we're at [0, 1, 1, 2, **2**, ~~3~~])

A. Add 2. $2+\frac{1}{3}\ =\frac{2}{1}+\frac{1}{3}=\frac{6}{3}+\frac{1}{3}\ =\frac{7}{3}$. (B. we're not done yet…)

C. The reciprocal of 7/3 is 3/7.

Move left; the next whole number is 2. ([0, 1, 1, **2**, ~~2, 3~~])

A. $2+\frac{3}{7}\ =\frac{2}{1}+\frac{3}{7}=\frac{14}{7}+\frac{3}{7}\ =\frac{17}{7}$. (B. I'll stop including this here; just remember to check every time!)

C. The reciprocal of 17/7 is 7/17

Move left; the next whole number is 1. ([0, 1, **1**, ~~2, 2, 3~~])

A. $1+\frac{7}{17}\ =\frac{17}{17}\ +\frac{7}{17}\ \ =\frac{24}{17}$.

C. The reciprocal of 24/17 is 17/24.

Move left the next whole number is 1. ([0, **1**, ~~1, 2, 2, 3~~])

A. $1+\frac{17}{24}\ =\frac{24}{24}+\frac{17}{24}=\frac{41}{24}$

C. The reciprocal of 41/24 is 24/41.

Move left; the next whole number is 0. ([**<u>0</u>**, ~~1, 1, 2, 2, 3~~]) (Note that this is the last one!)

A. $0 + 24/41 = 24/41$

B. 0 is the leftmost (i.e., last) whole number in the list, so we're done! Our result is $24/41$.

Let's compare… recall that $\log(1.5)/\log(2) = 0.5849625$… Also, recall that $7/12 = 0.5833333$…, which was off by about 0.0016. By comparison, $24/41 = 0.5853658$… . So, we have:

$$\frac{\log(1.5)}{\log(2)} = 0.5849625\ldots$$
$$\frac{24}{41} = 0.5853658\ldots$$

The difference between these two numbers is approximately 0.0004.

Thus, $24/41$ is a much closer approximation of $\log(1.5)/\log(2)$ than $7/12$. This translates into a possible **41-TET** system, under which **24** "semitones" approximates a perfect fifth.

Note: Another way to confirm that $24/41$ is a "better" approximation than $7/12$ is to compare 2^x for each of these "x" values, since that's what got us started on this problem in the first place:

$$2^{7/12} \approx 1.498307, \text{ and } 2^{24/41} \approx 1.500419.$$

The latter is much closer to 1.5 than the former (they're off by about 0.0017 and about 0.0004, respectively).

If we continue this process another couple of steps, we get the following results. (Note: try to duplicate the following lists of whole number parts on your own, following the steps laid out on page 2 above.)

Whole number list: [0, 1, 1, 2, 2, 3, 1]
Starting with this list, we end up with 31/53
(Check: $2^{31/53} \approx 1.4999409$. This is only "off" by 1.5 – 1.4999409 ≈ 0.0000591)

Whole number list: [0, 1, 1, 2, 2, 3, 1, 5]
Starting with this list, we end up with 179/306
(Check: $2^{179/306} \approx 1.5000050$. This time our error is about 1.5000050 – 1.5 = 0.000005)

(Those last two errors are approximately 6 *hundred-thousandths* and 5 *millionths*, respectively.)

Due to the above results, two of the more popular alternative tuning systems out there (for those who can make it work – e.g. electronic music composers) are 41-TET and 53-TET. The last result is not widely used, since 306 tones to the octave is rather cumbersome in any context.

EXAMPLE: Start with $\pi = 3.14159265 \ldots$ on your calculator.

1. Write down the whole number part, 3, to start our list.
2. Subtract 3, leaving us with 0.14159265…
3. Hit the reciprocal key, to get 7.0625133…

1. Write down the whole number part, 7. (Current list: [3, 7])
2. Subtract 7, leaving 0.062513305…
3. Hit the reciprocal key, to get 15.9965944…

1. Write round this off to 16, and then stop. (Current list: [3, 7, 16])

Let's get a few different approximations. First, we'll just use the first two "whole number parts," 3 and 7:

A. Add 7 (B. Not done yet)
C: The reciprocal of 7 is 1/7.

Move to the left; the next whole number is 3.
A. Add: $3 + \frac{1}{7} = \frac{3}{1} + \frac{1}{7} = \frac{21}{7} + \frac{1}{7} = \frac{22}{7}$
B. 3 is the leftmost whole number in the list, so we're done. **Result: $\mathbf{22/7}$**

Note: 22/7 is a very commonly used rational approximation for π. This is because it's very close to the actual value of π despite having a relatively "small" denominator of just 7. Here's the comparison:

$$\pi = 3.14159265 \ldots$$
$$\frac{22}{7} = 3.14285714 \ldots$$
$$\frac{22}{7} - \pi = 0.0012644 \ldots$$

We can get a "better" approximation (but with a much larger denominator, which is a trade-off) by using the full list [3, 7, 16]…

A. Add 16. (B. not finished yet, so continue)
C. The reciprocal of 15 is 1/16

Move to the left; the next whole number in the list is 7.
A. Add: 7 + 1/16 = 112/16 + 1/16 = 113/16 (B. not finished yet, so continue)
C: The reciprocal of 113/16 is 16/113

Move to the left; the next whole number is 3.
A. Add: 3 + 16/113 = 339/113 + 16/113 = 355/113
B: 3 is the leftmost whole number in the list, so we're done. Result: $\mathbf{355/113}$

Let's check: 333/106 = 3.1415094… Comparing this to π gives us $\pi - \frac{333}{106} = 0.00008321 \ldots$ (not bad!)

Practice Exercises: Rational Approximation

Use the method of "continued fractions" to find rational approximations for each of the following numbers:

1. log(50) 2. $\sqrt{5}$ 3. $\dfrac{\log(1.25)}{\log(2)}$

For each of these numbers, start by making a "whole number list" (as shown in the examples) consisting of at least five whole numbers. Then, find three different rational approximations for each number:

- find a rational approximation using three whole numbers (from your list)
- find a rational approximation using four whole numbers
- find a rational approximation using five whole numbers.

You should end up with three different rational approximations for each number, and they should get progressively better (i.e. closer to what you're trying to approximate) as you use more whole numbers. Make sure to check this for yourself – that is, each time you find a rational approximation, use your calculator to verify that what you are getting is "close" to the number being approximated.

Example:

Let's work through the first part of the first exercise, finding approximations of log(50). To get started, first find your "whole number list" for continued fractions for log(50). (This is where you alternately subtract the whole number part of what you see, then take the reciprocal of what's left.) You should get a list starting with [1, 1, 2, 3, 9]. For the first part, finding a "rational approximation using three whole numbers," you would use [1, 1, 2] from this list…

So, running through the second part of the process, we start with 2.

- Add 2
- The reciprocal of 2 is 1/2.

Moving right-to-left, the next whole number is 1.

- Add 1: $1+\frac{1}{2}=\frac{2}{2}+\frac{1}{2}=\frac{3}{2}$
- The reciprocal of 3/2 is 2/3.

Move right-to-left again; the next whole number is 1. (Note that we're at the end of our list!)

- Add 1: $1+\frac{2}{3}=\frac{3}{3}+\frac{2}{3}=\frac{5}{3}$

Since this is the last whole number in our list, we're done at this point! (No more reciprocals!)

Answer: 5/3.

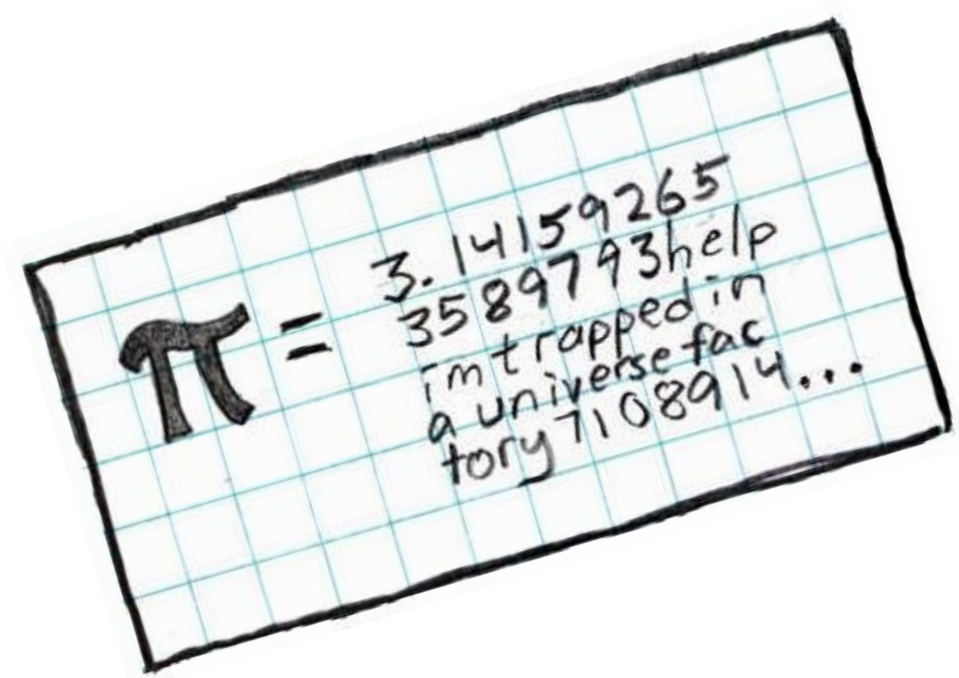
Check: 5/3 = 1.66666…., and log(50) = 1.69897…, so 5/3 is a pretty close estimate.
The "error" would be log(50)-5/3, which is about 0.03230.

Note: the above work can also be written using “continued fractions” notation, as follows:

$$\begin{aligned}
\log(50) &= \mathbf{1} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{2}.321\ldots}} \\
&\approx \mathbf{1} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{2}}} \\
&= 1 + \frac{1}{3/2} \\
&= 1 + \frac{2}{3} \\
&= \frac{5}{3}
\end{aligned}$$

One way of organizing the work is no better or worse than the other; go with what you’re the most comfortable with!

Answers to these practice exercises appear on the following page…



[Cartoon by Randall Munroe: https://xkcd.com/10]

Answers to practice exercises

1. log(50)

Your list of whole numbers should be [1, 1, 2, 3, 9]

Three whole numbers: [1, 1, 2]; rational approximation: 5/3
Four whole numbers: [1, 1, 2, 3]; rational approximation: 17/10
Five whole numbers: [1, 1, 2, 3, 9]; rational approximation: 158/93

2. $\sqrt{5}$

Your list of whole numbers should be [2, 4, 4, 4, 4]

Three whole numbers: [2, 4, 4]; rational approximation: 38/17
Four whole numbers: [2, 4, 4, 4]; rational approximation: 161/72
Five whole numbers: [2, 4, 4, 4, 4]; rational approximation: 682/305

3. $\log(1.25)/\log(2)$

Your list of whole numbers should be [0, 3, 9, 2, 2]

Three whole numbers: [0, 3, 9]; rational approximation: 9/28
Four whole numbers: [0, 3, 9, 2]; rational approximation: 19/59
Five whole numbers: [0, 3, 9, 2, 2]; rational approximation: 47/146