Section 1.3.1: Cents Measurement

In this section, we develop the notion of "cents" measurement. "Cents" is another method (besides frequency ratios) of measuring the "width" of the interval between two tones. One common application of "cents" is when one uses an electronic tuner; typically the user is given a number of "cents," between -50 and +50, telling her how much a tone deviates from a specified target pitch (e.g. A: 440). Simply put, a "cent" is one percent, or 1/100, of a 12-TET semitone. We'll give a more precise definition later in this section.

Before delving into cents measurement, let's briefly review and summarize what we know about interval widths (measured with frequency ratios) up to this point. The following table lists various intervals with their corresponding frequency ratios under various tuning systems. The leftmost column gives the music theory name for each interval; many of these should be familiar, but some may be new to you. The ratios listed in the just intonation, Pythagorean, and 12-TET columns of this table come directly from sections 1.1, 1.2, and 1.3, respectively.

Frequency Ratio Table

Interval name	# of semitones (standard keyboard)	Pythagorean frequency ratio	Just intonation frequency ratio	12-TET frequency ratio
minor second	1	256/243	16/15	$2^{1/12}$
major second	2	9/8	9/8	$2^{2/12}$ (or $2^{1/6}$)
minor third	3	32/27	6/5	$2^{3/12}$ (or $2^{1/4}$)
major third	4	81/64	5/4	$2^{4/12}$ (or $2^{1/3}$)
perfect fourth	5	4/3	4/3	$2^{5/12}$
diminished fifth	6	729/512	45/32	$2^{6/12}$ (or $2^{1/2}$)
perfect fifth	7	3/2	3/2	$2^{7/12}$
minor sixth	8	128/81	8/5	$2^{8/12}$ (or $2^{2/3}$)
major sixth	9	27/16	5/3	$2^{9/12}$ (or $2^{3/4}$)
minor seventh	10	16/9	9/5	$2^{10/12}$ (or $2^{5/6}$)
major seventh	11	243/128	15/8	$2^{11/12}$
octave	12	2	2	$2^{12/12}$ (or just 2!)

Cents Measurement

Under equal temperament, a "cent" is defined as one one-hundredth of a semitone. Thus, each semitone consists of 100 cents. If an octave consists of 12 semitones, then there are $100 \times 12 = 1200$ cents in an octave.

Imagine a keyboard with 1200 keys per octave – not very practical, but it's a way to imagine what a "cent" is. The purpose of "cents" is to measure very small differences in pitches and/or frequency ratios.

As is the case with semitones, cents are consistent under equal temperament – that is, two frequencies that are one cent apart always form an interval with the same frequency ratio. So, just as a semitone corresponds to a frequency ratio of $2^{1/12}$, a cent corresponds to a frequency ratio of $2^{1/1200}$.

The formula to convert from cents measurement to a frequency ratio is straightforward; the idea is identical to finding frequency ratio of an n-semitone interval in 12-TET. Recall that, since the frequency ratio of a semitone is $2^{1/12}$, it follows that the frequency ratio of an n-semitone interval is $2^{n/12}$. By the same reasoning: since the frequency ratio of a cent is $2^{1/1200}$, the frequency ratio of a c-cent interval is $2^{c/1200}$.

Thus, the formula to convert from cents (c) to frequency ratio (r) is

$$\boldsymbol{r = 2^{c/1200}}$$

<u>Example</u>: Suppose an interval is described as having a width of 240 cents. The frequency ratio of this interval would be $r = 2^{240/1200}$, which (use a calculator!) is approximately 1.149.

<u>Note</u>: another way to think of "cents" is as a fractional number of semitones. For example, 240 cents is the same thing as 2.4 semitones – that is, between two and three (but slightly closer to two) semitones on a standard piano keyboard. So, according to the 12-TET frequency ratio formula from the preceding section, we'd have the frequency ratio $r = 2^{2.4/12} \approx 1.1487$. Note that the exponent 2.4/12 is equivalent to the exponent 240/1200; the only distinction between these two equal fractions is a factor of 100 on the top and on the bottom.

<u>Example</u>: Lena plays a Bb (or A#) into a digital tuner. The display tells her that her Bb is 32 cents sharp – that is, 32 cents above the correct pitch for Bb. What is the frequency ratio between the correct pitch and the one Lena is playing?

<u>Answer</u>: The frequency ratio would be $r = 2^{32/1200} \approx 1.019$.

<u>Note</u>: In terms of frequency ratios, we could say that Lena's Bb is sharp by about 1.9%. (This comes from the .019 part of the frequency ratio – a frequency ratio of exactly 1 is an exact match, so the relative discrepancy is measured by 0.019, or 1.9%.)

We'd also like to be able to reverse this relationship – that is, to convert a given frequency ratio into cents measurement. To do so for an interval with frequency ratio r, we must find a number, *c*, such that

$$\boldsymbol{r = 2^{c/1200}}.$$

Since c is in the exponent of this equation, we must use logarithms to solve for it. (See section 1.3.2 for a brief review of logarithms, if necessary.) Using properties of logarithms, we can rewrite the above equation as follows:

$$\begin{aligned} r &= 2^{c/1200} \\ \log(r) &= \log\left(2^{\frac{c}{1200}}\right) \\ \log(r) &= \frac{c}{1200} \times \log(2) \\ 1200 \times \log(r) &= c \times \log(2) \\ 1200 \times \frac{\log(r)}{\log(2)} &= c \end{aligned}$$

Thus, the formula to convert from a frequency ratio (r) to cents (c) is =

$$\frac{\log(r)}{\log(2)} \times 1200.$$

Note: The quantity $\frac{\log(r)}{\log(2)}$ is the base-2 logarithm of r, which can also be written as $\log_2(r)$. This is defined as the exponent, x, such that $2^x = r$. If you have a calculator or computer that can find base-2 logarithms directly (the built-in Mac OS X calculator has a "$\log_2$" button, for example), then you can just use that rather than computing log(r)/log(2) using base-ten logarithms.

Example: A just intonation (or "ideal") major sixth has a frequency ratio of 5/3. To convert this to cents measurement, use the above formula:

$$c = \frac{\log\left(\frac{5}{3}\right)}{\log(2)} \times 1200 \approx 884\ cents$$

Comment: Recall that a major sixth is a nine-semitone interval. Does this result seem reasonable? What is the width in cents of a 12-TET major sixth? Is it wider or narrower than an ideal major sixth?

(Note: since a "cent" is a very small unit of measurement, it's usually sufficient to round off to the nearest whole number of cents.)

Exercise: Use the table of frequency ratios at the beginning of this section to find the measurement in "cents" of each of these intervals. Some of the answers are shown below, to give you the idea – check to make sure you can duplicate these results before working on the others. (The rest of the answers are provided on the next page.) Round each answer to the nearest whole number.

Intervals measured by "cents" (all rounded to the nearest cent)

Interval name	# of semitones (standard keyboard)	Pythagorean width in cents	Just intonation width in cents	12-TET width in cents
minor second	1	112		100
major second	2		204	
minor third	3	316		300
major third	4	386		
perfect fourth	5	498	498	500
diminished fifth	6		588	
perfect fifth	7	702	702	700
minor sixth	8			
major sixth	9		906	900
minor seventh	10	1018		
major seventh	11			
octave	12	1200	1200	1200

SOLUTIONS

Intervals measured by “cents” (all rounded to the nearest cent)
Based on the frequency ratio chart on the first page of this handout (copied below)

Interval name	# of semitones (standard keyboard)	Pythagorean width in cents	Just intonation width in cents	12-TET width in cents
minor second	1	90	112	100
major second	2	204	204	200
minor third	3	294	316	300
major third	4	407	386	400
perfect fourth	5	498	498	500
diminished fifth	6	611	590	600
perfect fifth	7	702	702	700
minor sixth	8	792	814	800
major sixth	9	906	884	900
minor seventh	10	996	1018	1000
major seventh	11	1110	1088	1100
octave	12	1200	1200	1200

Comment: Notice that the 12-TET intervals’ widths are all multiples of 100 cents. This is by design; remember that 12-TET is based on the premise of consistent semitones, and cents measurement defines each semitone to be exactly 100 cents wide.

Question: Which 12-TET intervals deviate the most (in terms of cents measurement) from their just intonation counterparts? Which intervals deviate the least? What problems, if any, might arise as a result of these discrepancies?

Frequency Ratio Table

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