

Certain Lambert Series Identities and their Proof via Trigonometry: Part 2

In the [last post](#) we saw that using a trigonometric identity Ramanujan was able to express the functions $S_{2n-1}(x)$ or equivalently $\Phi_{0,2n-1}(x)$ in terms of simpler functions $P(x), Q(x), R(x)$. Continuing our journey further we start with the equation:

$$\begin{aligned} & \left(\frac{1}{4} \cot \frac{\theta}{2} + \frac{x \sin \theta}{1-x} + \frac{x^2 \sin 2\theta}{1-x^2} + \frac{x^3 \sin 3\theta}{1-x^3} + \dots \right)^2 \\ &= \left(\frac{1}{4} \cot \frac{\theta}{2} \right)^2 + \frac{x \cos \theta}{(1-x)^2} + \frac{x^2 \cos 2\theta}{(1-x^2)^2} + \frac{x^3 \cos 3\theta}{(1-x^3)^2} + \dots \\ &+ \frac{1}{2} \left\{ \frac{x(1-\cos \theta)}{1-x} + \frac{2x^2(1-\cos 2\theta)}{1-x^2} + \frac{3x^3(1-\cos 3\theta)}{1-x^3} + \dots \right\} \end{aligned} \quad (1)$$

which was established in the [last post](#).

Expression of $\Phi_{1,2n}(x)$ in terms of P, Q, R

Using the above identity we will now express the functions $\Phi_{1,2n}(x)$ in terms of P, Q, R . To do so we need to express both the LHS and RHS of (1) in series of powers of θ . From the last post we can see that

$$\begin{aligned} \frac{1}{4} \cot \frac{\theta}{2} &= \frac{1}{2\theta} - \frac{B_2}{2 \cdot 2} \cdot \frac{\theta}{1!} + \frac{B_4}{2 \cdot 4} \cdot \frac{\theta^3}{3!} - \dots \\ &= \frac{1}{2\theta} + E_1 \frac{\theta}{1!} - E_3 \frac{\theta^3}{3!} + E_5 \frac{\theta^5}{5!} - \dots \\ \frac{x^m}{1-x^m} \sin m\theta &= \frac{x^m}{1-x^m} \left(m\theta - \frac{m^3 \theta^3}{3!} + \frac{m^5 \theta^5}{5!} - \dots \right) \end{aligned}$$

and therefore the LHS of (1) can be written as:

$$\left(\frac{1}{2\theta} + \frac{\theta}{1!} S_1(x) - \frac{\theta^3}{3!} S_3(x) + \frac{\theta^5}{5!} S_5(x) - \dots \right)^2$$

For the RHS of (1) we can see that

$$\begin{aligned} \left(\frac{1}{4} \cot \frac{\theta}{2} \right)^2 &= \frac{1}{16} \cot^2 \frac{\theta}{2} \\ &= \frac{1}{4\theta^2} - \frac{1}{24} + \frac{1}{2} \left(- \frac{B_4}{2 \cdot 4} \frac{\theta^2}{2!} + \frac{B_6}{2 \cdot 6} \frac{\theta^4}{4!} - \frac{B_8}{2 \cdot 8} \frac{\theta^6}{6!} + \dots \right) \\ &= \frac{1}{4\theta^2} + E_1 + \frac{1}{2} \left(E_3 \frac{\theta^2}{2!} - E_5 \frac{\theta^4}{4!} + \dots \right) \end{aligned}$$

and

$$\begin{aligned} \frac{mx^m(1-\cos m\theta)}{1-x^m} &= \frac{mx^m}{1-x^m} \left(\frac{m^2 \theta^2}{2!} - \frac{m^4 \theta^4}{4!} + \dots \right) \\ \frac{x^m \cos m\theta}{(1-x^m)^2} &= \frac{x^m}{(1-x^m)^2} \left(1 - \frac{m^2 \theta^2}{2!} + \frac{m^4 \theta^4}{4!} - \dots \right) \end{aligned}$$

Noting that

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{m^r x^m}{(1-x^m)^2} &= \sum_{m=1}^{\infty} m^r x^m \sum_{n=1}^{\infty} n x^{mn-m} \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^r n x^{mn} = \Phi_{1,r}(x)\end{aligned}$$

the RHS of equation (1) can be written as:

$$\begin{aligned}\frac{1}{4\theta^2} + E_1 + \Phi_{1,0}(x) - \frac{\theta^2}{2!} \Phi_{1,2}(x) + \frac{\theta^4}{4!} \Phi_{1,4}(x) - \frac{\theta^6}{6!} \Phi_{1,6}(x) + \dots \\ + \frac{1}{2} \left(\frac{\theta^2}{2!} S_3(x) - \frac{\theta^4}{4!} S_5(x) + \dots \right)\end{aligned}$$

Thus the equation (1) is transformed into:

$$\begin{aligned}\left(\frac{1}{2\theta} + \frac{\theta}{1!} S_1(x) - \frac{\theta^3}{3!} S_3(x) + \frac{\theta^5}{5!} S_5(x) - \dots \right)^2 \\ = \frac{1}{4\theta^2} + S_1(x) - \frac{\theta^2}{2!} \Phi_{1,2}(x) + \frac{\theta^4}{4!} \Phi_{1,4}(x) - \frac{\theta^6}{6!} \Phi_{1,6}(x) + \dots \\ + \frac{1}{2} \left(\frac{\theta^2}{2!} S_3(x) - \frac{\theta^4}{4!} S_5(x) + \dots \right)\end{aligned}$$

For even positive integer n if we equate the coefficients of θ^n on both sides of the above equation we get:

$$\begin{aligned}\frac{n+3}{2(n+1)} S_{n+1}(x) - \Phi_{1,n}(x) &= \binom{n}{1} S_1(x) S_{n-1}(x) \\ &\quad + \binom{n}{3} S_3(x) S_{n-3}(x) + \dots \\ &\quad + \binom{n}{n-1} S_{n-1}(x) S_1(x)\end{aligned}\tag{2}$$

Putting $n = 2$ we get

$$\begin{aligned}\frac{5}{6} S_3(x) - \Phi_{1,2}(x) &= 2 S_1^2(x) \\ \Rightarrow \frac{5}{6} \frac{Q}{240} - \Phi_{1,2}(x) &= 2 \left(\frac{-P}{24} \right)^2\end{aligned}$$

or

$$288 \Phi_{1,2}(x) = Q - P^2\tag{3}$$

Similarly with $n = 4, 6$ we get

$$720 \Phi_{1,4}(x) = PQ - R\tag{4}$$

$$1008 \Phi_{1,6}(x) = Q^2 - PR\tag{5}$$

Now that we have expressed $\Phi_{1,2n}(x)$ in terms of P, Q, R we proceed to handle the case for general $\Phi_{r,s}(x)$.

Expression of $\Phi_{r,s}(x)$ in terms of P, Q, R

We know that

$$P = 1 - 24\Phi_{1,0}(x) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)x^n$$

and therefore

$$x \frac{dP}{dx} = -24 \sum_{n=1}^{\infty} n\sigma_1(n)x^n = -24\Phi_{1,2}(x)$$

and using (3) we get

$$x \frac{dP}{dx} = \frac{P^2 - Q}{12} \quad (6)$$

Similarly we obtain:

$$x \frac{dQ}{dx} = \frac{PQ - R}{3} \quad (7)$$

$$x \frac{dR}{dx} = \frac{PR - Q^2}{2} \quad (8)$$

Again it is easy to verify (via repeated differentiation) that

$$\Phi_{r,s}(x) = \left(x \frac{d}{dx} \right)^r \Phi_{0,s-r}(x) \quad (9)$$

If $s - r$ is odd (i.e. r and s are of different parity) then $\Phi_{0,s-r}(x)$ can be expressed as a polynomial in P, Q, R and thereby using (6), (7), (8) we can see from the above equation that $\Phi_{r,s}(x)$ can be expressed as a polynomial in P, Q, R .

We show one calculation for $\Phi_{2,3}(x)$. Clearly from (9) we have

$$\begin{aligned} \Phi_{2,3}(x) &= \left(x \frac{d}{dx} \right)^2 \Phi_{0,1}(x) \\ &= x \frac{d}{dx} \Phi_{1,2}(x) \\ &= x \frac{d}{dx} \left(\frac{Q - P^2}{288} \right) \\ &= \frac{x}{288} \left(\frac{dQ}{dx} - 2P \frac{dP}{dx} \right) \\ &= \frac{1}{288} \left(\frac{PQ - R}{3} - \frac{P^3 - PQ}{6} \right) \\ &= \frac{3PQ - 2R - P^3}{1728} \end{aligned}$$

Using equations (6), (7), (8) we can see that

$$\begin{aligned} x \frac{d}{dx} (Q^3 - R^2) &= 3Q^2 x \frac{dQ}{dx} - 2Rx \frac{dR}{dx} \\ &= 3Q^2 \frac{PQ - R}{3} - 2R \frac{PR - Q^2}{2} \\ &= PQ^3 - RQ^2 - PR^2 + RQ^2 \\ &= P(Q^3 - R^2) \end{aligned}$$

and hence it follows that

$$x \frac{d}{dx} \{\log(Q^3 - R^2)\} = P \quad (10)$$

Again if we note the definition of eta function $\eta(x)$ given by

$$\eta(x) = x^{1/24}(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots$$

it is clear that

$$P = 24x \frac{d}{dx} \{\log(\eta(x))\}$$

or

$$P = x \frac{d}{dx} \{\log(\eta^{24}(x))\} \quad (11)$$

From (10), (11) it follows that we must have

$$Q^3 - R^2 = A\eta^{24}(x)$$

where A is some constant. Since we have

$$Q = 1 + 240x + \cdots, \quad R = 1 - 504x + \cdots$$

it follows that

$$Q^3 - R^2 = (3 \cdot 240 + 2 \cdot 504)x + \cdots = 1728x + \cdots$$

and therefore the constant A must be 1728.

We finally arrive at the beautiful identity:

$$Q^3 - R^2 = 1728x \{(1-x)(1-x^2)(1-x^3)\cdots\}^{24} \quad (12)$$

Relation of P, Q, R with Elliptic Integrals K, E

Using the differential equations (6), (7), (8) we can deduce many further properties of P, Q, R . Here we will establish their link with the elliptic integrals K, E and modulus k . We know that

$$\frac{dK}{dk} = \frac{E - k'^2 K}{kk'^2}, \quad \frac{dE}{dk} = \frac{E - K}{k}$$

Eliminating E from the above relations we get

$$kk'^2 \frac{d^2 K}{dk^2} + (1 - 3k^2) \frac{dK}{dk} - kK = 0 \quad (13)$$

Now if we set $x = q^2$ we find that

$$\begin{aligned}
 P(q^2) &= 12q \frac{d}{dq} [\log\{q^{1/12}(1-q^2)(1-q^4)(1-q^6)\cdots\}] \\
 &= 12q \frac{dk}{dq} \frac{d}{dk} \left\{ \log\left(2^{-1/3} \sqrt{\frac{2K}{\pi}} (kk')^{1/6}\right) \right\} \\
 &= \frac{24kk'^2 K^2}{\pi^2} \left\{ \frac{1}{2K} \frac{dK}{dk} + \frac{1}{6k} - \frac{k}{6k'^2} \right\} \\
 &= \left(\frac{2K}{\pi}\right)^2 \left\{ \frac{3kk'^2}{K} \cdot \frac{E - k'^2 K}{kk'^2} + k'^2 - k^2 \right\} \\
 &= \left(\frac{2K}{\pi}\right)^2 \left\{ \frac{3E}{K} + k^2 - 2 \right\}
 \end{aligned} \tag{14}$$

From (6) and (14) we see that

$$\begin{aligned}
 q \frac{dP(q^2)}{dq} &= \frac{P^2(q^2) - Q(q^2)}{6} \\
 \Rightarrow Q(q^2) &= P^2(q^2) - 6q \frac{dP(q^2)}{dq} \\
 &= \left(\frac{2K}{\pi}\right)^4 \left\{ \frac{3E}{K} + k^2 - 2 \right\}^2 - 6q \frac{dk}{dq} \frac{d}{dk} \{P(q^2)\} \\
 &= \left(\frac{2K}{\pi}\right)^4 \left[\left\{ \frac{3kk'^2}{K} \frac{dK}{dk} + 1 - 2k^2 \right\}^2 \right. \\
 &\quad \left. - \frac{3kk'^2}{K^2} \frac{d}{dk} \left\{ 3kk'^2 K \frac{dK}{dk} + K^2(1 - 2k^2) \right\} \right] \\
 &= \left(\frac{2K}{\pi}\right)^4 \left[\left\{ \frac{3kk'^2}{K} \frac{dK}{dk} + 1 - 2k^2 \right\}^2 \right. \\
 &\quad \left. - \frac{3kk'^2}{K^2} \left\{ 3kk'^2 \left(K \frac{d^2K}{dk^2} + \left(\frac{dK}{dk} \right)^2 \right) \right. \right. \\
 &\quad \left. \left. + K \frac{dK}{dk} (5 - 13k^2) - 4kK^2 \right\} \right] \\
 &= \left(\frac{2K}{\pi}\right)^4 \left[\left\{ \frac{3kk'^2}{K} \frac{dK}{dk} + 1 - 2k^2 \right\}^2 \right. \\
 &\quad \left. - \frac{9k^2 k'^4}{K} \frac{d^2K}{dk^2} - \frac{9k^2 k'^4}{K^2} \left(\frac{dK}{dk} \right)^2 \right. \\
 &\quad \left. - \frac{3kk'^2}{K} (5 - 13k^2) \frac{dK}{dk} + 12k^2 k'^2 \right] \\
 &= \left(\frac{2K}{\pi}\right)^4 \left\{ -\frac{9k^2 k'^4}{K} \frac{d^2K}{dk^2} \right. \\
 &\quad \left. + \frac{9kk'^2}{K} (3k^2 - 1) \frac{dK}{dk} + (1 - 2k^2)^2 + 12k^2 k'^2 \right\} \\
 &= \left(\frac{2K}{\pi}\right)^4 \left\{ -\frac{9kk'^2}{K} \left(kk'^2 \frac{d^2K}{dk^2} + (1 - 3k^2) \frac{dK}{dk} - kK \right) \right. \\
 &\quad \left. + 1 - 4k^2 + 4k^4 + 3k^2 - 3k^4 \right\} \\
 &= \left(\frac{2K}{\pi}\right)^4 (1 - k^2 + k^4)
 \end{aligned} \tag{15}$$

Similarly using (7), (14), (15) we get

$$\begin{aligned}
 R(q^2) &= P(q^2)Q(q^2) - \frac{3q}{2} \frac{dQ(q^2)}{dq} \\
 &= \left(\frac{2K}{\pi} \right)^6 \left\{ \frac{3kk'^2}{K} \frac{dK}{dk} + 1 - 2k^2 \right\} \{1 - k^2k'^2\} \\
 &\quad - \frac{3kk'^2K^2}{\pi^2} \frac{d}{dk} \left\{ \left(\frac{2K}{\pi} \right)^4 \{1 - k^2k'^2\} \right\} \\
 &= \left(\frac{2K}{\pi} \right)^6 \left[\left\{ \frac{3kk'^2}{K} \frac{dK}{dk} + 1 - 2k^2 \right\} \{1 - k^2k'^2\} \right. \\
 &\quad \left. - \frac{3kk'^2}{4K^4} \frac{d}{dk} \{K^4 \{1 - k^2k'^2\}\} \right] \\
 &= \left(\frac{2K}{\pi} \right)^6 \left[\left\{ \frac{3kk'^2}{K} \frac{dK}{dk} + 1 - 2k^2 \right\} \{1 - k^2k'^2\} \right. \\
 &\quad \left. - \frac{3kk'^2}{4K^4} \left\{ K^4(4k^3 - 2k) + 4K^3 \frac{dK}{dk}(1 - k^2k'^2) \right\} \right] \\
 &= \left(\frac{2K}{\pi} \right)^6 \left[\left\{ \frac{3kk'^2}{K} \frac{dK}{dk} + 1 - 2k^2 \right\} \{1 - k^2k'^2\} \right. \\
 &\quad \left. + \frac{3k^2k'^2}{2} (1 - 2k^2) - \frac{3kk'^2}{K} \frac{dK}{dk} (1 - k^2k'^2) \right] \\
 &= \left(\frac{2K}{\pi} \right)^6 (1 - 2k^2) \left(1 + \frac{k^2k'^2}{2} \right) \\
 &= \left(\frac{2K}{\pi} \right)^6 (1 + k^2)(1 - 2k^2) \left(1 - \frac{k^2}{2} \right)
 \end{aligned} \tag{16}$$

Going further we can replace q^2 by q in (15), (16) and note that this leads to replacing K by $(1+k)K$ and k by $2\sqrt{k}/(1+k)$ (see [Landen's Transformation of second order](#)). When this is done we get:

$$\begin{aligned}
 Q(q) &= \left(\frac{2(1+k)K}{\pi} \right)^4 \left(1 - \frac{4k}{(1+k)^2} + \frac{16k^2}{(1+k)^4} \right) \\
 &= \left(\frac{2K}{\pi} \right)^4 \{(1+k)^4 - 4k(1+k)^2 + 16k^2\} \\
 &= \left(\frac{2K}{\pi} \right)^4 \{(1+k)^2(1-k)^2 + 16k^2\} \\
 &= \left(\frac{2K}{\pi} \right)^4 (1 + 14k^2 + k^4)
 \end{aligned} \tag{17}$$

and

$$\begin{aligned}
 R(q) &= \left(\frac{2(1+k)K}{\pi} \right)^6 \left(1 + \frac{4k}{(1+k)^2} \right) \\
 &\quad \left(1 - \frac{8k}{(1+k)^2} \right) \left(1 - \frac{2k}{(1+k)^2} \right) \\
 &= \left(\frac{2K}{\pi} \right)^6 \{(1+k^2+6k)(1+k^2-6k)(1+k^2)\} \\
 &= \left(\frac{2K}{\pi} \right)^6 \{((1+k^2)^2 - 36k^2)(1+k^2)\} \\
 &= \left(\frac{2K}{\pi} \right)^6 (1+k^2)(1-34k^2+k^4)
 \end{aligned} \tag{18}$$

To get $P(q)$ we need to first see how $\eta(q^2)$ transforms when q^2 is replaced by q . Clearly we have

$$\eta(q^2) = q^{1/12}(1 - q^2)(1 - q^4)(1 - q^6) \cdots = 2^{-1/3} \sqrt{\frac{2K}{\pi}} (kk')^{1/6}$$

so that

$$\begin{aligned} \eta(q) &= q^{1/24}(1 - q)(1 - q^2)(1 - q^3) \cdots \\ &= 2^{-1/3} \sqrt{\frac{2(1+k)K}{\pi}} \left(\frac{2\sqrt{k}}{1+k} \sqrt{1 - \frac{4k}{(1+k)^2}} \right)^{1/6} \\ &= 2^{-1/6} \sqrt{\frac{2K}{\pi}} k^{1/12} k'^{1/3} \end{aligned} \quad (19)$$

and then we can see that

$$\begin{aligned} P(q) &= 24q \frac{d}{dq} \{\log \eta(q)\} \\ &= \frac{48kk'^2K^2}{\pi^2} \frac{d}{dk} \left\{ \log \left(2^{-1/6} \sqrt{\frac{2K}{\pi}} k^{1/12} k'^{1/3} \right) \right\} \\ &= 12kk'^2 \left(\frac{2K}{\pi} \right)^2 \left(\frac{1}{2K} \frac{dK}{dk} + \frac{1}{12k} - \frac{k}{3k'^2} \right) \\ &= \left(\frac{2K}{\pi} \right)^2 \left(\frac{6kk'^2}{K} \frac{dK}{dk} + 1 - 5k^2 \right) \\ &= \left(\frac{2K}{\pi} \right)^2 \left(\frac{6E}{K} + k^2 - 5 \right) \end{aligned} \quad (20)$$

Another set of formulas come by replacing q with $-q$ in (17), (18), (20). Note that replacing q with $-q$ leads to replacing K with $k'K$ and k^2 with $(-k^2/k'^2)$. With this understanding we have:

$$\begin{aligned} Q(-q) &= \left(\frac{2k'K}{\pi} \right)^4 \left(1 - \frac{14k^2}{k'^2} + \frac{k^4}{k'^4} \right) \\ &= \left(\frac{2K}{\pi} \right)^4 (k'^4 - 14k^2k'^2 + k^4) \\ &= \left(\frac{2K}{\pi} \right)^4 \{(1 - k^2)^2 - 14k^2(1 - k^2) + k^4\} \\ &= \left(\frac{2K}{\pi} \right)^4 (1 - 16k^2 + 16k^4) \\ &= \left(\frac{2K}{\pi} \right)^4 (1 - 16k^2k'^2) \\ &= \left(\frac{2K}{\pi} \right)^4 (1 - 4G^{-24}) \end{aligned} \quad (21)$$

and

$$\begin{aligned}
 R(-q) &= \left(\frac{2k'K}{\pi} \right)^6 \left(1 - \frac{k^2}{k'^2} \right) \left(1 + 34 \frac{k^2}{k'^2} + \frac{k^4}{k'^4} \right) \\
 &= \left(\frac{2K}{\pi} \right)^6 (1 - 2k^2)(k'^4 + 34k^2k'^2 + k^4) \\
 &= \left(\frac{2K}{\pi} \right)^6 (1 - 2k^2)(1 + 32k^2k'^2) \\
 &= \left(\frac{2K}{\pi} \right)^6 (1 - 2k^2)(1 + 8G^{-24})
 \end{aligned} \tag{22}$$

To get $P(-q)$ requires more work and to that end we first rewrite the relation (20) in another form:

$$\begin{aligned}
 P(q) &= \left(\frac{2K}{\pi} \right)^2 \left(\frac{6kk'^2}{K} \frac{dK}{dk} + 1 - 5k^2 \right) \\
 &= \left(\frac{2K}{\pi} \right)^2 \left(\frac{12k^2k'^2}{2kK} \frac{dK}{dk} + 1 - 5k^2 \right) \\
 &= \left(\frac{2K}{\pi} \right)^2 \left(\frac{12k^2k'^2}{K} \frac{dK}{d(k^2)} + 1 - 5k^2 \right)
 \end{aligned}$$

and then we can find $P(-q)$ as follows:

$$\begin{aligned}
 P(-q) &= \left(\frac{2k'K}{\pi} \right)^2 \left\{ \frac{-12k^2}{k'^2} \left(1 + \frac{k^2}{k'^2} \right) \frac{1}{k'K} \frac{d(k'K)}{d(-k^2/k'^2)} + 1 + \frac{5k^2}{k'^2} \right\} \\
 &= \left(\frac{2K}{\pi} \right)^2 \left(\frac{6kk'}{K} \frac{d(k'K)}{dk} + 1 + 4k^2 \right) \\
 &= \left(\frac{2K}{\pi} \right)^2 \left(\frac{6kk'^2}{K} \frac{dK}{dk} + 1 - 2k^2 \right) \\
 &= \left(\frac{2K}{\pi} \right)^2 \left(\frac{6(E - k'^2K)}{K} + 1 - 2k^2 \right) \\
 &= \left(\frac{2K}{\pi} \right)^2 \left(\frac{6E}{K} + 4k^2 - 5 \right)
 \end{aligned} \tag{23}$$

We next define another invariant $J(q)$ by:

$$J(q) = \frac{Q^3(-q) - R^2(-q)}{Q^3(-q)} = \frac{1728\eta^{24}(-q)}{Q^3(-q)}$$

Now from (19) we see that

$$\eta^{24}(q) = 2^{-4} \left(\frac{2K}{\pi} \right)^{12} k^2 (1 - k^2)^4$$

and hence replacing q by $-q$ we get

$$\begin{aligned}
 \eta^{24}(-q) &= 2^{-4} \left(\frac{2k'K}{\pi} \right)^{12} \cdot \frac{-k^2}{k'^2} \left(1 + \frac{k^2}{k'^2} \right)^4 \\
 &= 2^{-4} \left(\frac{2K}{\pi} \right)^{12} (-k^2k'^2) \\
 &= -2^{-6} \left(\frac{2K}{\pi} \right)^{12} G^{-24}
 \end{aligned}$$

Using (21) and definition of $J(q)$ above we arrive at

$$J(q) = \frac{Q^3(-q) - R^2(-q)}{Q^3(-q)} = \frac{-27G^{-24}}{(1 - 4G^{-24})^3} = \frac{-27G^{48}}{(G^{24} - 4)^3} \quad (24)$$

Also using expressions for $\eta^{24}(-q), Q(-q), R(-q)$ in terms of G we can establish that $Q^3(-q) - R^2(-q) = 1728\eta^{24}(-q)$ thereby furnishing another proof of (12).

Identities Concerning Divisor Functions

From the last post we have the following identities:

$$1 + 480\Phi_{0,7}(x) = Q^2 \quad (25)$$

$$1 - 264\Phi_{0,9}(x) = QR \quad (26)$$

$$691 + 65520\Phi_{0,11}(x) = 441Q^3 + 250R^2 \quad (27)$$

$$1 - 24\Phi_{0,13}(x) = Q^2R \quad (28)$$

First of these identities can be written as:

$$1 + 480 \sum_{n=1}^{\infty} \sigma_7(n)x^n = \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)x^n \right)^2$$

Equating coefficients of x^n on both sides we get:

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{i=1}^{n-1} \sigma_3(i)\sigma_3(n-i)$$

so that $\sigma_7(n) - \sigma_3(n)$ is divisible by 120 for all positive integers n .

Similarly from (26) we get:

$$11\sigma_9(n) = \{21\sigma_5(n) - 10\sigma_3(n)\} + 5040 \sum_{i=1}^{n-1} \sigma_3(i)\sigma_5(n-i)$$

and thus $11\sigma_9(n) + 10\sigma_3(n) - 21\sigma_5(n)$ is divisible by 5040.

From (25) and (28) we get

$$1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n)x^n = \left(1 + 480 \sum_{n=1}^{\infty} \sigma_7(n)x^n \right) \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)x^n \right)$$

and thus on equating coefficients of x^n we get

$$\sigma_{13}(n) = \{21\sigma_5(n) - 20\sigma_7(n)\} + 10080 \sum_{i=1}^{n-1} \sigma_5(i)\sigma_7(n-i)$$

We can rewrite the equation (27) as

$$691 + 65520\Phi_{0,11}(x) = 441(Q^3 - R^2) + 691R^2$$

so that

$$691 + 65520\Phi_{0,11}(x) = 441 \cdot 1728\eta^{24}(x) + 691R^2 \quad (29)$$

Let us define Ramanujan's Tau function $\tau(n)$ by

$$\sum_{n=1}^{\infty} \tau(n)x^n = \eta^{24}(x) = x \prod_{n=1}^{\infty} (1 - x^n)^{24}$$

and then equating coefficients of x^n in (29) we get

$$\begin{aligned} 65520\sigma_{11}(n) &= 441 \cdot 1728\tau(n) - 691 \cdot 1008\sigma_5(n) \\ &\quad + 691 \cdot 504^2 \sum_{i=1}^{n-1} \sigma_5(i)\sigma_5(n-i) \\ \Rightarrow 65\sigma_{11}(n) &= 756\tau(n) - 691\sigma_5(n) + 691 \cdot 252 \sum_{i=1}^{n-1} \sigma_5(i)\sigma_5(n-i) \\ \Rightarrow 65(\sigma_{11}(n) - \tau(n)) &= 691\tau(n) - 691\sigma_5(n) + 691 \cdot 252 \sum_{i=1}^{n-1} \sigma_5(i)\sigma_5(n-i) \end{aligned}$$

and thus we arrive at the famous congruence satisfied by $\tau(n)$

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691} \quad (30)$$

By Paramanand Singh
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Labels: Lambert Series , Mathematical Analysis , Trigonometry

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