Section 1.3.2 Review of Exponents and Logarithms

Note: This document is not intended or designed to teach you about exponents and logarithms from scratch. It's assumed that you've seen all of this at some point in the past, and that this section is just a refresher.

For each of the following expressions, b represents a positive real number.

DEFINITION: If n is a counting number – i.e., 1, 2, 3, 4, … – then

$$b^n = \underbrace{b \cdot b \cdot \ldots \cdot b}_{multiply\ n\ times}$$

That is, b^n is shorthand for repeatedly multiplying by b a total of n times.

EXAMPLES

$$4^7 = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$$

$$(1.06)^4 = 1.06 \cdot 1.06 \cdot 1.06 \cdot 1.06$$

$\left(\frac{3}{2}\right)^{12} = \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2}$ (which may also be written as $\frac{3^{12}}{2^{12}}$).

RULES FOR EXPONENTS: For any real numbers m and n:

1. Product Rule:

$$b^m \cdot b^n = b^{m+n}$$

2. Quotient Rule:

$$\frac{b^m}{b^n} = b^{m-n}$$

3. Power-of-a-power Rule:

$$(b^m)^n = b^{mn}$$

Examples:

1. $4^3 \cdot 4^5 = \underbrace{\left(\underbrace{4 \cdot 4 \cdot 4}_{4^3}\right)\left(\underbrace{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}_{4^5}\right)}_{Multiply\ by\ 4\ \ 3+5=8\ \ times\ in\ all} = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$, or 4^8

2. $\frac{3^9}{3^5} = \underbrace{\frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}}_{\substack{Cancel\ five\ 3's \\ on\ top\ and\ bottom}} = \underbrace{\frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1}{1 \cdot 1 \cdot 1 \cdot 1 \cdot 1}}_{\substack{After\ cancellation, \\ left\ with\ 9-5=4 \\ 3's\ remaining\ on\ top}} = 3 \cdot 3 \cdot 3 \cdot 3$, or 3^4

3. $((1.5)^3)^3 = \left(\underbrace{1.5 \cdot 1.5 \cdot 1.5}_{this\ is\ (1.5)^3}\right)^3 = \underbrace{(1.5 \cdot 1.5 \cdot 1.5)(1.5 \cdot 1.5 \cdot 1.5)(1.5 \cdot 1.5 \cdot 1.5)}_{Multiply\ by\ 1.5\ \ 3\times 3=9\ \ times\ in\ all.} = (1.5)^9$

Based on the above rules, we may adopt the following conventions for negative and fractional exponents…

- $b^0 = 1$ (for any positive number b)
- If n is positive, then $b^{-n} = \frac{1}{b^n}$. (e.g. $3^{-4} = \frac{1}{3^4}$)

  (This is because $3^{-4} = 3^{0-4}$, which by the Quotient Rule is equal to $\frac{3^0}{3^4}$, which equals $\frac{1}{3^4}$)
- If n is any number other than zero, then $b^{1/n} = \sqrt[n]{b}$. (e.g. $2^{1/12} = \sqrt[12]{2}$)

  (This is because of the Power-of-a-power rule: for example, $\left(2^{1/12}\right)^{12} = 2^{\left(\frac{1}{12}\cdot\frac{12}{1}\right)} = 2^1$, which means $2^{1/12}$ is the twelfth root of 2.)

These conventions allow us to define powers where the exponent is any fraction.
For example: $2^{7/12} = \left(2^{1/12}\right)^7$ -- that is, the twelfth root of 2, raised to the 7th power.

NOTE: This is convenient for us, since under 12-TET, all frequency ratios can be written as powers of the twelfth root of 2; this is why we can write the frequency ratio of any n-semitone interval as $2^{n/12}$.

<u>**LOGARITHMS**</u>

<u>DEFINITION</u>: $\log_b(x)$ is a function that "reverses" the exponential function b^x. That is, if you tell me the value of b^x, then I can use the "log" function to tell you the value of the exponent x.

Examples:
$10^3 = 1000$. The base is 10, and the exponent is 3; therefore, $\log_{10}(1000) = 3$.
$2^5 = 32$. The base is 2, and the exponent is 5; therefore, $\log_2(32) = 5$.
$9^{3/2} = 27$. The base is 9, and the exponent is 3/2; therefore, $\log_9(27) = \frac{3}{2}$.

NOTE: Your calculator has a button labeled "log". This actually calculates the function "$\log_{10}$" – that is, <u>base ten</u> logarithms. For example, enter the number 1000, then hit "log". (Or type in "log(1000)" if that's how your calculator works.) Compare the result to the first example above. This is what the "log" button does!

Why we need logarithms:
Sometimes we need to be able to solve an equation like the following:

$$2^x = 1.5$$

This is called an "exponential equation," since the unknown quantity (x) is in the exponent. There is no way to use arithmetic to solve for x; in fact, usually the solution of an exponential equation will be an irrational number rather than a whole number or a fraction. We could use trial-and-error to guess the value of x; after a while we might narrow it down to within a few thousandths. However, this would be time consuming, tedious, and completely unnecessary.

Since LOGARITHMS "reverse" exponential functions (such as 2^x), they can help us to find solutions to exponential equations quickly.

<u>COMPUTING LOGARITHMS OF OTHER BASES</u> (here "other" means "other than 10")

To solve an exponential equation with a base other than 10, use the following equation:

$$\mathbf{log_{10}}(\boldsymbol{b^x}) = \boldsymbol{x} \cdot \mathbf{log_{10}}(\boldsymbol{b})$$

WHY THIS WORKS:
As we mentioned earlier, $\log_{10}(x)$ "reverses" 10^x; that is, these are "inverse functions" of one another.
Let a stand for the value of $\log_{10}(b)$. In other words, $10^a = b$. (For example, if $b = 1000$, then $a = 3$.)
With this in mind, look at the left-hand side of the equation $\mathbf{log_{10}}(\boldsymbol{b^x}) = \boldsymbol{x} \cdot \mathbf{log_{10}}(\boldsymbol{b})$.
We can rewrite the part inside the parentheses, b^x, as follows:

$$b^x \underbrace{\qquad = \qquad}_{\substack{\text{Substitute } 10^a \\ \text{in place of } b \\ \text{at this step...}}} (10^a)^x \underbrace{\qquad = \qquad}_{\substack{\text{Use "power-of-a-power"} \\ \text{rule at this step...}}} 10^{ax}$$

So, b^x is the same thing as 10^{ax}. Therefore, $\log_{10}(b^x)$ is the same thing as $\log_{10}(10^{ax})$. Now, because of the definition of logarithm, the "log" of 10^{ax} is simply the exponent, ax. Thus, the left-hand side of the equation at the top of this page, $\log_{10}(b^x)$, can be replaced with ax.

Now, look at the right-hand side of the equation at the top of this page. Recall that we defined the number a as being equal to $\log_{10}(b)$, so we can substitute one for the other. So, the right-hand side of the equation, $x \cdot \log_{10}(b^x)$, can be replaced with ax.

Therefore, $\log_{10}(b^x)$ and $x \cdot \log_{10}(b)$ are both equal to ax, which means they are equal to each other. Thus, the formula $\mathbf{log_{10}}(\boldsymbol{b^x}) = \boldsymbol{x} \cdot \mathbf{log_{10}}(\boldsymbol{b})$ always works! (Note that the reasoning behind this is valid for *every* number x, and for *every* positive number b.)

Now, here's why this matters for us. Let's go back to the equation $2^x = 1.5$. Applying the above rule to both sides of this equation gives us:

$$\log_{10}(2^x) = \log_{10}(1.5)$$
$$x \cdot \log_{10}(2) = \log_{10}(1.5)$$

At this point, we can solve for x by dividing both sides of the equation by $\log_{10}(2)$:

$$x = \frac{\log_{10}(1.5)}{\log_{10}(2)},$$

which can be approximated on a calculator: $x \approx 0.5849625007$.

(Check with a calculator: $2^{0.5849625007} \approx 1.5$, which is what we wanted!)

Change of Base Formula

We could also "solve" the equation $2^x = 1.5$ by rewriting the exponential function (whose base is 2, rather than 10) as a logarithmic function (again with base 2, rather than 10). That would look like this: $x = \log_2(1.5)$. This has the benefit of being correct, but unfortunately calculators don't have a "$\log_2$" button. However, the solution we found above using base ten logarithms must be the same as the one using base two logarithms; that is,

$$\log_2(1.5) = \frac{\log_{10}(1.5)}{\log_{10}(2)}.$$

See where the 1.5 and the 2 end up on either side? Something like this always works; that is, for *any* logarithm of the form $\log_b(a)$, we can *always* use the following formula to estimate that logarithm on a calculator:

$$\textbf{"Change of Base" Formula}: \log_b(a) = \frac{\log_{10}(b)}{\log_{10}(a)}.$$

This formula works for any two positive numbers a and b.

Exercises:

1. Use either the formula from "computing logarithms of other bases" subsection (along with a little bit of algebra) or the "change of base" formula to solve each of the following equations. For each, find an exact value (in terms of logarithms) for x, and also give a decimal approximation (rounded to three decimal places).

a. $2^x = 10$
b. $4^x = 100$
c. $3^x = 256$
d. $(1.5)^x = 128$

2. Simplify each of the following, using properties of exponents. You should be able to write each answer as a whole number.

a) $2^9 \times 2^{-4}$ b) $\left(5^{1/3}\right)^9$ c) $16^{3/2}$ d) $27^{4/3}$ e) $\frac{\left(6^{80} \times 6^{45}\right)}{6^{123}}$

Answers appear on the following page:

Answers to Exercises on previous page:

1. a) $\frac{\log(10)}{\log(2)} = \frac{1}{\log(2)} \approx 3.322$ b) $\frac{\log(100)}{\log(4)} = \frac{2}{\log(4)} \approx 3.322$

c) $\frac{\log(256)}{\log(3)} \approx 5.047$ d) $\frac{\log(128)}{\log(1.5)} \approx 11.967$

2. a) $2^5 = 32$ b) $5^3 = 125$ c) $4^3 = 64$ d) $3^4 = 81$ e) $6^2 = 36$