Values of Rogers-Ramanujan Continued Fraction: Part 1

A Wild Theorem by Ramanujan

In his letter dated 16th January 1913 to G. H. Hardy, Ramanujan presented the following *wild* theorem:

$$\cfrac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{1+\cdots}}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right)\sqrt[5]{e^{2\pi}} \tag{1}$$

The theorem looks so strange and surprising, coming out of nowhere that Hardy had to remark: *"they must be true because, if they were not true, no one would have had the imagination to invent them."* In this post we will prove the above theorem using elementary methods. The proof is essentially the one given by Watson who claimed that probably Ramanujan obtained the result in the same manner.

Rogers-Ramanujan Continued Fraction

We have encountered Rogers-Ramanujan Continued Fraction in an earlier post, defined by

$$R(q) = \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}} = \frac{q^{1/5}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\cdots \tag{2}$$

Using this definition of $R(q)$ we can easily see that the result (1) above can be expressed in simpler notation as:

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}$$

so that Ramanujan provided evaluation of the function $R(q)$ for a certain value of q namely $q = e^{-2\pi}$. In fact Ramanujan gave many other evaluations of $R(q)$ and also a general formula for $R(q)$ in terms of the class invariants. We will however only focus on the evaluation of $R(e^{-2\pi})$ and $R(-e^{-\pi})$ in this post.

We have established earlier that the function $R(q)$ satisfies the following identity

$$R(q) = q^{1/5}\cdot\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = q^{1/5}\frac{H(q)}{G(q)}$$

where the functions $G(q), H(q)$ satisfy the following Rogers-Ramanujan identities:

$$\begin{aligned}G(q) &= \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}\\&= 1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \cdots\\&= 1 + \sum_{n=1}^{\infty}\frac{q^{n^2}}{(q;q)_n}\\H(q) &= \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}\\&= 1 + \frac{q^2}{1-q} + \frac{q^6}{(1-q)(1-q^2)} + \frac{q^{12}}{(1-q)(1-q^2)(1-q^3)} + \cdots\\&= 1 + \sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}\end{aligned}$$

Ramanujan found many other identities satisfied by $R(q)$ the simplest of which is the following:

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \tag{3}$$

where $f(-q) = (q;q)_\infty$. It will be seen that this identity is fundamental in the evaluation of $R(e^{-2\pi})$. We now turn to the proof of this identity.

Proof of the Fundamental Identity Concerning $R(q)$

Since we know from the Euler's Pentagonal Theorem that

$$f(-q) = (q;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2}$$

it follows that the RHS of equation (3) can be written as

$$\frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} = \frac{\displaystyle\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/10}}{\displaystyle q^{1/5}\sum_{n=-\infty}^{\infty}(-1)^n q^{5n(3n+1)/2}} \tag{4}$$

First Stage

We first need to simplify RHS of equation (4). The numbers $n(3n+1)$ are all even and if we reduce them modulo 10 we can see that they will take the values $0, 2, 4$ and therefore the expression $n(3n+1)/10$ can be expressed as $a_n + (i/5)$ where a_n is some integer and i can take values $0, 1, 2$. Also it should be noted that:

$$\begin{aligned}i = 0 &\Leftrightarrow n \equiv 0, 3 \pmod 5\\i = 1 &\Leftrightarrow n \equiv 4 \pmod 5\\i = 2 &\Leftrightarrow n \equiv 1, 2 \pmod 5\end{aligned}$$

Thus we may write

$$f(-q^{1/5}) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/10} = J_1 + q^{2/5}J_2 + q^{1/5}J_3$$

where J_1, J_2, J_3 are power series in q with coefficients ± 1. Again putting $n = 5m + 4$ we can see that

$$\begin{aligned} q^{1/5}J_3 &= \sum_{m=-\infty}^{\infty}(-1)^{5m+4}q^{(5m+4)(15m+13)/10}\\ &= \sum_{m=-\infty}^{\infty}(-1)^{m}q^{(5m+4)(15m+13)/10}\\ &= \sum_{m=-\infty}^{\infty}(-1)^{m-1}q^{(5m-1)(15m-2)/10}\\ &= -\sum_{m=-\infty}^{\infty}(-1)^{m}q^{1/5}q^{5m(3m-1)/2}\\ &= -q^{1/5}\sum_{m=-\infty}^{\infty}(-1)^{m}q^{5m(3m-1)/2}\\ &= -q^{1/5}\sum_{m=-\infty}^{\infty}(-1)^{m}q^{5m(3m+1)/2}\\ &= -q^{1/5}f(-q^5) \qquad (5)\end{aligned}$$

It now follows that

$$\begin{aligned}\frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} &= \frac{J_1+q^{2/5}J_2+q^{1/5}J_3}{-q^{1/5}J_3}\\ &= q^{-1/5}J_1'-1+q^{1/5}J_2' \qquad (6)\end{aligned}$$

where $J_1' = -J_1/J_3, J_2' = -J_2/J_3$ are power series of q. Now we know that $J_3 = -f(-q^5)$ therefore

$$J_1' = \frac{1}{(q^5;q^5)_\infty}\cdot\sum_{n\equiv 0,3 \pmod 5}(-1)^n q^{n(3n+1)/10} \qquad (7)$$

Second Stage

The identity (3) concerning $R(q)$ can be now established if we can prove the following two identities:

$$q^{-1/5}J_1' = \frac{1}{R(q)} \qquad (8)$$

$$q^{1/5}J_2' = -R(q) \qquad (9)$$

If we multiply the above two identities we can see that $J_1'J_2' = -1$ and hence if we prove that

$$J_1'J_2' = -1 \qquad (10)$$

then we need to establish only one of the identities (8) and (9).

Establishing (10) is bit tricky but not difficult. It is achieved by cubing the identity (6) and doing further careful analysis. First we can rewrite (6) in the form

$$\frac{f(-q^{1/5})}{f(-q^5)} = J_1' - q^{1/5} + q^{2/5}J_2' \qquad (11)$$

Clearly upon cubing (11) we see that the LHS becomes

$$\begin{aligned}\left(\frac{f(-q^{1/5})}{f(-q^{5})}\right)^{3} &= \frac{\displaystyle\sum_{n=0}^{\infty}(-1)^{n}(2n+1)q^{n(n+1)/10}}{\displaystyle\sum_{n=0}^{\infty}(-1)^{n}(2n+1)q^{5n(n+1)/2}}\\ &= \frac{A_1+q^{1/5}A_2+q^{3/5}A_3}{\displaystyle\sum_{n=0}^{\infty}(-1)^{n}(2n+1)q^{5n(n+1)/2}}\end{aligned}$$

where

$$\begin{aligned}A_1 &= \sum_{n\geq 0, n\equiv 0,4 \pmod 5}(-1)^{n}(2n+1)q^{n(n+1)/10}\\ q^{1/5}A_2 &= \sum_{n\geq 0, n\equiv 1,3 \pmod 5}(-1)^{n}(2n+1)q^{n(n+1)/10}\\ q^{3/5}A_3 &= \sum_{n\geq 0, n\equiv 2 \pmod 5}(-1)^{n}(2n+1)q^{n(n+1)/10}\end{aligned}$$

so that A_1, A_2, A_3 are power series in q.

Thus we have now

$$\left(\frac{f(-q^{1/5})}{f(-q^{5})}\right)^{3} = A'_1+q^{1/5}A'_2+q^{3/5}A'_3 \tag{12}$$

where A'_1, A'_2, A'_3 are power series in q.

The cube of RHS of equation (11) can be written as

$$\begin{aligned}(J'_1-q^{1/5}+q^{2/5}J'_2)^3 &= (J'_1-q^{1/5}+q^{2/5}J'_2)(J'_1-q^{1/5}+q^{2/5}J'_2)^2\\ &= (J'_1-q^{1/5}+q^{2/5}J'_2)\\ &\quad (J'^2_1+q^{2/5}+q^{4/5}J'^2_2\\ &\quad -2q^{1/5}J'_1-2q^{3/5}J'_2+2q^{2/5}J'_1J'_2)\\ &= (J'^3_1+q^{2/5}J'_1+q^{4/5}J'_1J'^2_2\\ &\quad -2q^{1/5}J'^2_1-2q^{3/5}J'_1J'_2\\ &\quad +2q^{2/5}J'^2_1J'_2-q^{1/5}J'^2_1-q^{3/5}\\ &\quad -qJ'^2_2+2q^{2/5}J'_1+2q^{4/5}J'_2\\ &\quad -2q^{3/5}J'_1J'_2+q^{2/5}J'^2_1J'_2+q^{4/5}J'_2\\ &\quad +qq^{1/5}J'^3_2-2q^{3/5}J'_1J'_2\\ &\quad -2qJ'^2_2+2q^{4/5}J'_1J'^2_2)\\ &= (J'^3_1-3qJ'^2_2)-q^{1/5}(3J'^2_1-qJ'^3_2)\\ &\quad -q^{3/5}(1+6J'_1J'_2)+3q^{2/5}J'_1(1+J'_1J'_2)\\ &\quad +3q^{4/5}J'_2(1+J'_1J'_2)\end{aligned} \tag{13}$$

From equations (11), (12) and (13) we can see that we must have

$$\begin{aligned}3q^{2/5}J'_1(1+J'_1J'_2)+3q^{4/5}J'_2(1+J'_1J'_2) &= 0\\ \Rightarrow (1+J'_1J'_2)(q^{2/5}J'_1+q^{4/5}J'_2) &= 0\end{aligned}$$

Clearly we can't have $q^{2/5}J_1' + q^{4/5}J_2'$ equal to zero unless both J_1', J_2' are zero identically. Hence it follows that we have $1 + J_1'J_2' = 0$ identically for all q. Thus equation (10) is established.

Third Stage

We next need to establish equation (8). Clearly this is equivalent to proving that

$$J_1' = \frac{(q^2; q^5)_\infty (q^3; q^5)_\infty}{(q; q^5)_\infty (q^4; q^5)_\infty}$$

Now from equation (7) we can see that

$$(q^5; q^5)_\infty J_1' = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(15n+1)/2} + \sum_{n=-\infty}^{\infty} (-1)^n q^{(5n-2)(3n-1)/2} \tag{14}$$

The Quintuple Product Identity states that

$$(q; q)_\infty (qz; q)_\infty (1/z; q)_\infty (qz^2; q^2)_\infty (q/z^2; q^2)_\infty = \sum_{n=-\infty}^{\infty} q^{(3n^2+n)/2}(z^{3n} - z^{-3n-1})$$

Replacing q by q^5 and z by $-1/q^4$ we get

$$\begin{aligned}
&(q^5; q^5)_\infty (-q; q^5)_\infty (-q^4; q^5)_\infty (q^{-3}; q^{10})_\infty (q^{13}; q^{10})_\infty \\
&= \sum_{n=-\infty}^{\infty} q^{5n(3n+1)/2}\{(-1)^{3n} q^{-12n} - (-1)^{-3n-1} q^{12n+4}\} \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{5n(3n+1)/2}(q^{-12n} + q^{12n+4})
\end{aligned}$$

A little manipulation with the term $(q^{-3}; q^{10})_\infty$ leads to

$$\begin{aligned}
&(q^5; q^5)_\infty (-q; q^5)_\infty (-q^4; q^5)_\infty (q^3; q^{10})_\infty (q^7; q^{10})_\infty \\
&= \sum_{n=-\infty}^{\infty} (-1)^{n+1} q^{5n(3n+1)/2}(q^{-12n+3} + q^{12n+7})
\end{aligned} \tag{15}$$

Replacing n by $n-1$ the RHS above becomes

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} (-1)^n q^{(5n-5)(3n-2)/2}(q^{-12n+15} + q^{12n-5}) \\
&= \sum_{n=-\infty}^{\infty} (-1)^n \{q^{(15n^2-49n+40)/2} + q^{(15n^2-n)/2}\}
\end{aligned}$$

Replacing n by $n+2$ in the first sum we see that the RHS of equation (15) becomes

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} (-1)^n \{q^{(15n^2+11n+2)/2} + q^{(15n^2-n)/2}\} \\
&= \sum_{n=-\infty}^{\infty} (-1)^n \{q^{(15n^2-11n+2)/2} + q^{(15n^2+n)/2}\} \\
&= \sum_{n=-\infty}^{\infty} (-1)^n \{q^{(5n-2)(3n-1)/2} + q^{n(15n+1)/2}\}
\end{aligned}$$

so that it matches the RHS of equation (14).

It now follows that the LHS of equation (14) matches that of equation (15) and therefore

$$\begin{aligned}(q^5;q^5)_\infty J_1' &= (q^5;q^5)_\infty(-q;q^5)_\infty(-q^4;q^5)_\infty(q^3;q^{10})_\infty(q^7;q^{10})_\infty\\ \Rightarrow J_1' &= (-q;q^5)_\infty(-q^4;q^5)_\infty(q^3;q^{10})_\infty(q^7;q^{10})_\infty\\ &= \frac{(q^2;q^{10})_\infty(q^8;q^{10})_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}\cdot(q^3;q^{10})_\infty(q^7;q^{10})_\infty\\ &= \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}\end{aligned}$$

and thus the equation (8) is established.

We have thus proved the fundamental identity satisfied by $R(q)$ namely

$$\frac{1}{R(q)}-1-R(q)=\frac{f(-q^{1/5})}{q^{1/5}f(-q^5)}$$

Evaluation of $R(e^{-2\pi})$

Putting $q=e^{-2\pi}$ in equation (3) and setting $x=R(e^{-2\pi})$ we get

$$\frac{1}{x}-1-x=\frac{f(-e^{-2\pi/5})}{e^{-2\pi/5}f(-e^{-10\pi})} \tag{16}$$

If we use the definition of eta function from an earlier post

$$\eta(q)=q^{1/12}f(-q^2)$$

and its transformation formula from the same post given by

$$\frac{\eta(e^{-\pi\sqrt{n}})}{\eta(e^{-\pi/\sqrt{n}})}=n^{-1/4}$$

for any $n>0$, then we can see that by putting $n=25$ we get

$$\begin{aligned}&\frac{\eta(e^{-5\pi})}{\eta(e^{-\pi/5})}=\frac{1}{\sqrt{5}}\\ &\Rightarrow \frac{e^{-5\pi/12}f(-e^{-10\pi})}{e^{-\pi/60}f(-e^{-2\pi/5})}=\frac{1}{\sqrt{5}}\\ &\Rightarrow \frac{e^{-2\pi/5}f(-e^{-10\pi})}{f(-e^{-2\pi/5})}=\frac{1}{\sqrt{5}}\end{aligned}$$

Now using (16) we get

$$\begin{aligned}&\frac{1}{x}-1-x=\sqrt{5}\\ &\Rightarrow x^2+(\sqrt{5}+1)x-1=0\\ &\Rightarrow x=\frac{-(\sqrt{5}+1)\pm\sqrt{10+2\sqrt{5}}}{2}\end{aligned}$$

Since $x>0$ we must take $+$ sign in the above value of x and then we get

$$x = R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}$$

and thus the *wild* theorem (1) given by Ramanujan is established.

Evaluation of $R(-e^{-\pi})$

If we substitute $q = -e^{-\pi}$ in equation (3) and let $x = R(-e^{-\pi})$ we get

$$\frac{1}{x} - 1 - x = -\frac{f(e^{-\pi/5})}{e^{-\pi/5} f(e^{-5\pi})} \tag{17}$$

Now if we take the definition of eta function from this post, namely $\eta(q) = q^{1/24} f(-q)$ then we have the transformation formula

$$\frac{\eta^{24}(-e^{-\pi\sqrt{n}})}{\eta^{24}(-e^{-\pi/\sqrt{n}})} = n^{-6}$$

Putting $n = 25$ we get

$$\begin{aligned}
&\frac{\eta^{24}(-e^{-5\pi})}{\eta^{24}(-e^{-\pi/5})} = 25^{-6}\\
&\Rightarrow \frac{-e^{-5\pi} f^{24}(e^{-5\pi})}{-e^{-\pi/5} f^{24}(e^{-\pi/5})} = 25^{-6}\\
&\Rightarrow \frac{e^{-\pi/5} f(e^{-5\pi})}{f(e^{-\pi/5})} = 25^{-1/4} = \frac{1}{\sqrt{5}}
\end{aligned}$$

Using equation (17) we get

$$\begin{aligned}
&\frac{1}{x} - 1 - x = -\sqrt{5}\\
&\Rightarrow x^2 + (1 - \sqrt{5})x - 1 = 0\\
&\Rightarrow x = \frac{\sqrt{5} - 1 \pm \sqrt{10 - 2\sqrt{5}}}{2}
\end{aligned}$$

Since $x < 0$ we must take $-$ sign above and then we get

$$x = R(-e^{-\pi}) = \frac{\sqrt{5}-1}{2} - \sqrt{\frac{5-\sqrt{5}}{2}}$$

In Ramanujan's letter to Hardy this result is written as another wild theorem

$$\cfrac{1}{1 - \cfrac{e^{-\pi}}{1 + \cfrac{e^{-2\pi}}{1 - \cfrac{e^{-3\pi}}{1 + \cdots}}}} = \left(\sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}\right)\sqrt[5]{e^{\pi}} \tag{18}$$

By Paramanand Singh
Sunday, September 8, 2013

Labels: Continued Fractions , Mathematical Analysis , Theta Functions