The Riemann Integral: Part 3

Oscillation of a Function

In a previous post we obtained the Riemann's condition of integrability using the concept of upper and lower Darboux sums. We observed that in order that a function be Riemann integrable on interval $[a, b]$ it was necessary (and sufficient) to make the sum

$$U(P, f) - L(P, f) = \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1})$$

arbitrarily small for some partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a, b]$.

Now it is obvious that this sum can be made arbitrarily smaller if both the parts $(M_k - m_k)$ and $(x_k - x_{k-1})$ can be made arbitrarily smaller. Making the $\Delta x_k = x_k - x_{k-1}$ small represents no hurdle, we just need to choose partitions of sufficiently small norms.

It is the difference $M_k - m_k$ which requires further probing. Technically this difference is called the *oscillation of function f on interval* $[x_{k-1}, x_k]$. More formally we define the *oscillation of a function f on an interval I (denoted by $\Omega_f(I)$) as follows:*

$$\Omega_f(I) = \sup\{f(x) - f(y) \mid x, y \in I\}$$

Clearly $\Omega_f(I)$ exists and is non-negative provided f is bounded in I. It should be obvious that $\Omega_f(I_1) \leq \Omega_f(I_2)$ whenever $I_1 \subseteq I_2$. It also makes sense to define the oscillation of a function f at a certain point. Let f be defined in a certain neighborhood of a point a. Then we define the *oscillation of f at a (denoted by $\omega_f(a)$) as follows:*

$$\omega_f(a) = \lim_{h \to 0+} \Omega_f((a - h, a + h))$$

The above limit always exists (provided f is bounded in neighborhood of a) because when $h \to 0+$ the oscillation decreases and is always non-negative. It should also be obvious that $\omega_f(a) = 0$ if and only if the function f is continuous at a.

We next need to understand that if the oscillation of a function at a certain point a is small then its oscillation is also small at points in the neighborhood of a. To be precise we have the following property:
If for some given $\epsilon > 0$ we have $\omega_f(a) < \epsilon$ then there is a neighborhood I of a such that $\omega_f(x) < \epsilon$ for all $x \in I$.

To see why this is the case we need to understand that if $\omega_f(a) = A < \epsilon$ then there is a neighborhood I of a such that $\Omega_f(I) < (A + \epsilon)/2$. Since I is an open interval, if $x \in I$ then we can find a neighborhood of x which is wholly contained in I and in this neighborhood the oscillation of f does not exceed $(A + \epsilon)/2$ and hence the oscillation $\omega_f(x) \leq (A + \epsilon)/2 < \epsilon$.

From the above it follows that:
If f is bounded in $[a, b]$ then for any given $\epsilon > 0$ the set of points

$$I_\epsilon = \{x \mid x \in [a, b], \omega_f(x) < \epsilon\}$$

is open.
and therefore it is quite obvious that:
If f is bounded on $[a, b]$ then for any given $\epsilon > 0$ the set of points

$$J_\epsilon = \{x \mid x \in [a, b], \omega_f(x) \geq \epsilon\}$$

is closed.

Also we need another result (whose proof is reminiscent of the proof for uniform continuity) on oscillation of a function:
If a function f is bounded on $[a, b]$ and there is an $\epsilon > 0$ such that $\omega_f(x) < \epsilon$ for all $x \in [a, b]$ then we can find a partition of $[a, b]$ such that the oscillation of f in each of the sub-intervals generated by this partition is less then ϵ.

Clearly if $\omega_f(x) < \epsilon$ then there is a neighborhood I_x of x in which the oscillation of f is less than ϵ. Together all these neighborhoods I_x form an open cover for $[a, b]$ and hence by Heine-Borel Principle there is a finite set of neighborhoods I_x which covers the interval $[a, b]$. Naturally these neighborhoods must overlap and the end points of these neighborhoods partition the interval $[a, b]$ into a finite number of sub-intervals. Let δ be a positive number less than the length of least of these sub-intervals. Then any sub-interval of length less than δ is contained in some interval I_x and hence the oscillation of f in this sub-interval is less than ϵ. We thus only need to form a partition of $[a, b]$ whose norm is less than δ and then oscillation of f in each of the sub-intervals generated by P is less than ϵ.

Condition for Integrability

Let a function f be bounded in $[a, b]$ and let $\sigma > 0$ be given. If we consider a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ then there will be certain sub-intervals $[x_{k-1}, x_k]$ in which the oscillation of f is less than σ and in the remaining sub-intervals the oscillation of f will be greater than or equal to σ. Then the sum

$$U(P, f) - L(P, f) = \sum_{k=1}^{n}(M_k - m_k)\Delta x_k$$

can be split into two sums S_1, S_2 where S_1 is based on sub-intervals where the oscillation of f is less than σ and the sum S_2 is based on the remaining sub-interval. Clearly then we have

$$S_1 \leq \sigma(b - a),\ \ S_2 \geq \sigma \sum \Delta x_k = \sigma L_\sigma$$

where the sum $L_\sigma = \sum \Delta x_k$ in bove equation is based on the sub-intervals where the oscillation of f is greater than or equal to σ. If f is integrable on $[a, b]$ then we must be able to make the difference $U(P, f) - L(P, f)$ less than ϵ for any given $\epsilon > 0$. Clearly this would mean that $S_2 < \epsilon$ so that $\sigma L_\sigma \leq S_2 < \epsilon$. Thus we have $L_\sigma < \epsilon/\sigma$.

So we see that in order that a function be integrable on an interval it is necessary that given any $\sigma > 0$ we must be able to make the sum of lengths of sub-intervals where oscillation of f is greater or equal to σ arbitrarily small. Riemann understood that this condition is also sufficient for integrability. Let $\epsilon > 0$ be given. Also let $M = \Omega_f([a, b])$. We set $\sigma = \epsilon/(2(b-a))$ and we choose a partition P such that the sum of lengths of sub-intervals where the oscillation of f is greater than or equal to σ is less than $\epsilon/2M$. Clearly then we have

$$S_1 < \frac{\epsilon}{2(b-a)} \cdot (b-a) = \frac{\epsilon}{2},\ S_2 < M \cdot \frac{\epsilon}{2M} = \frac{\epsilon}{2}$$

so that $U(P, f) - L(P, f) < \epsilon$. What we have established above is the following:

Let f be a function bounded on $[a, b]$. Then f is integrable on $[a, b]$ if and only if for any given $\sigma > 0, \lambda > 0$ it is possible to find a partition P of $[a, b]$ such that the sum of lengths of sub-intervals of P where the oscillation of f exceeds σ is less than λ.

Lebesgue's Criterion of Integrability

In the above condition of integrability by Riemann we are forced to consider the lengths of sub-intervals satisfying some particular property. Namely we want to constrain the length of sub-intervals where the oscillation of the function is not small. We also know that if a function is continuous at a point then its oscillation at that point is zero (and conversely). Therefore the sub-intervals where the oscillation is not small should contain points of discontinuity of the function. So in effect Riemann condition implies that we should be able to cover the points of discontinuity by a set of intervals whose total length is arbitrarily small. This important idea was first formalized by Henri Lebesgue and was given the name of *"measure"*.

More formally we say that *a set of points is of measure zero if for any given $\epsilon > 0$ it can be covered by a countable collection of open intervals the total sum of whose lengths is less than ϵ.*

It is trivial that a set consisting of a single point is of measure zero. Also if we have a countable number of sets S_i each of which is of measure zero, then their union is also of measure zero. Clearly given $\epsilon > 0$ we can cover the set S_i by a countable collection of intervals whose total length is less than $\epsilon/2^i$ and hence the union of all sets S_i can be covered by a countable collection of intervals whose total length is less than $\sum(\epsilon/2^i) = \epsilon$. It is thus clear that any countable set is of measure zero and in particular the set of all rational numbers is of measure zero. There also exist uncountable sets which are of measure zero and we shall indicate one famous example of such a set later.

So Riemann condition seems to suggest that the set of discontinuities of an integrable function should be of measure zero. This condition was established by Lebesgue and can be stated as follows:

Let f be a function bounded on $[a, b]$. Then f is Riemann integrable on $[a, b]$ if and only if the *set of discontinuities of f in $[a, b]$ is of measure zero.*

It is easy to observe that the set of discontinuities of f can be expressed as a union

$$D = \bigcup_{i=1}^{\infty} D_i$$

where

$$D_i = \left\{x \mid x \in [a, b],\, \omega_f(x) \geq \frac{1}{i}\right\}$$

If D is not of measure zero then there is some set D_i which is not of measure zero. Clearly in that case there exists an $\epsilon > 0$ such that D_i can not be covered by any collection of intervals whose total length is less than ϵ. If P is a partition of $[a, b]$ we can split the difference $U(P, f) - L(P, f) = \sum(M_k - m_k)(x_k - x_{k-1})$ into two sums S_1, S_2 such that S_1 is based on those sub-intervals which don't contain any point of D_i. Then clearly sub-intervals in S_2 form a cover for the points of D_i and hence in each of these sub-intervals the oscillation of f is greater than or equal to $1/i$. Also the total sum of the lengths of these sub-intervals is not less than ϵ. Therefore it follows that $U(P, f) - L(P, f) \geq S_2 \geq \epsilon/i$ for any partition P of $[a, b]$. Clearly this means that the function f is not Riemann integrable on $[a, b]$.

Next we prove that if set D is of measure zero then f is integrable on $[a, b]$. This is bit subtle in terms of exact formal argument and the reader must pay attention to what follows. Since D is of measure zero and $D_i \subseteq D$ for all i, it follows that each D_i is of measure zero. Hence for each i the set D_i can be covered by a countable collection of open intervals the sum of whose lengths is less than $1/i$. Since the set D_i is closed and bounded, Heine-Borel Principle applies here (this although requires a stronger form of the principle than we have established in previous post). Therefore set D_i can be covered by a finite number of open intervals the sum of whose lengths is less than $1/i$. Let the union of these open intervals be denoted by A_i and let B_i denote set of points which are in $[a, b]$ but not in A_i. Then clearly B_i is made up of a finite number of closed intervals. If I is one of the intervals making up B_i then we have $\omega_f(x) < 1/i$ for all $x \in I$. Hence the interval I can be further partitioned into sub-intervals such that oscillation of f in each of these sub-intervals in less than $1/i$. Thus we observe that the set B_i can be partitioned into sub-intervals such that the oscillation of f in each of these sub-intervals is less than $1/i$. The end points of these sub-intervals make a partition P of $[a, b]$. We then express the difference

$$U(P, f) - L(P, f) = \sum(M_k - m_k)(x_k - x_{k-1}) = S_1 + S_2$$

where S_1 is based on those sub-intervals which contain points of D_i and S_2 is based on the remaining sub-intervals. Clearly then we see that sub-intervals corresponding to S_1 are covered by A_i and here $M_k - m_k \leq M - m$ and $\sum \Delta x_k < 1/i$ so that $S_1 < (M - m)/i$. The sub-intervals corresponding to S_2 are contained in B_i and here we have $M_k - m_k < 1/i$ and $\sum \Delta x_k \leq (b - a)$ so that $S_2 < (b - a)/i$. It follows that

$$U(P, f) - L(P, f) < (M - m + b - a)/i$$

Thus corresponding to any positive integer i we have a partition P such that

$$U(P,f) - L(P,f) < (M - m + b - a)/i$$

It follows that f is integrable on $[a,b]$.

If a property holds on a certain set of points A except for a set of points of measure zero then we say that the property holds *almost everywhere* (abbreviated frequently as a.e.) on set A. Thus the Lebesgue's criterion can be expressed concisely as follows:
A bounded function f is integrable on $[a,b]$ if and only if it is continuous almost everywhere in $[a,b]$.

This result can be considered as the beginning of the theory of *Measure and Integration* as developed by Henri Lebesgue. We shall have occasion to deal with the rich and elegant theory developed by Lebesgue in later posts. For now we shall be content to give an example of an uncountable set which is of measure zero.

The Cantor Set

Let $A_0 = [0,1]$. We remove the middle third open interval $(1/3, 2/3)$ from A_0 to get $A_1 = [0,1/3] \cup [2/3,1]$. Thus A_1 is composed of 2 intervals whose total length is $1 - 1/3 = 2/3$. Now we remove the middle third open interval from each of these two intervals to get A_2 which consists now of 4 closed intervals whose total length is $1 - 1/3 - 2/9 = 4/9$. We proceed in this similar fashion indefinitely to generate sets A_i which consists of 2^i closed intervals whose total length is

$$1 - \left(\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots + \frac{2^{i-1}}{3^i}\right) = \left(\frac{2}{3}\right)^i$$

Let

$$A = \bigcap_{i=0}^{\infty} A_i$$

It is obvious that $0 \in A, 1 \in A$ and hence set A is non-empty. The set A is called Cantor set.

Clearly A can be covered by A_i whose length is $(2/3)^i$ and $(2/3)^i \to 0$ as $i \to \infty$, it follows that set A is of measure zero. It needs to be shown that set A is uncountable. If we observe the construction of set A deeply we will see that it consists of those numbers of interval $[0,1]$ whose ternary expansion (as opposed to decimal expansion) consists of only 0 or 2 but not 1. Such numbers can't be arranged in a sequence. If they could be arranged in sequence say $x_1, x_2, \ldots$ then we could write

$$\begin{aligned} x_1 &= 0.x_{11}x_{12}x_{13}\ldots \\ x_2 &= 0.x_{21}x_{22}x_{23}\ldots \\ x_3 &= 0.x_{31}x_{32}x_{33}\ldots \\ \ldots \end{aligned}$$

where the representation is in ternary and each of x_{ij} is either 0 or 2. Clearly we can now construct a number $y = 0.y_1y_2y_3\dots$ such that $y_n = 2 - x_{nn}$. Then clearly $y_n \neq x_{nn}$ for any n. Thus clearly y belongs to A, but does not belong to sequence $\{x_n\}$. This famous technique of Cantor called *diagonal slash* shows that the set A is uncountable.

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