The Riemann Integral: Part 2

In the last post we defined the Riemann integral of a function on a closed interval and discussed some of the conditions for the integrability of a function. Here we develop the full machinery of the Riemann integral starting with the basic properties first.

Basic Properties of the Riemann Integral

The following properties of the Riemann integral are not too difficult to establish:
If f and g are Riemann integrable over $[a, b]$ then so are the following functions:

- *$p \cdot f + q \cdot g$ for any arbitrary real numbers p, q and*
$$\int_{a}^{b} (pf(x) + qg(x))\,dx = p\int_{a}^{b} f(x)\,dx + q\int_{a}^{b} g(x)\,dx$$
- $|f|$
- f^2
- $f \cdot g$
- *f/g provided g is bounded away from zero in $[a, b]$ i.e. if there exists an $m > 0$ such that $|g(x)| > m$ in $[a, b]$*

The first property tells us that the operation of Riemann integration is linear. The linearity is obvious from the way a Riemann sum is defined. To check for the integrability of $|f|$ we need only note that if M_k, m_k are supremum and infimum of f in $[x_{k-1}, x_k]$ then $|M_k|, |m_k|$ are the supremum and infimum (but not necessarily in corresponding order) of $|f|$ in $[x_{k-1}, x_k]$. And since $||M_k| - |m_k|| \leq (M_k - m_k)$ it follows that

$$U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f)$$

Therefore integrability of f implies the integrability of $|f|$ on the same interval.

To prove the integrability of f^2 we note that $f^2 = |f|^2$ and clearly we have shown above that $|f|$ is integrable. Hence it is safe to assume that f is non-negative on $[a, b]$. If M_k, m_k are supremum and infimum of f in $[x_{k-1}, x_k]$ then M_k^2, m_k^2 are the supremum and infimum (in corresponding order) of f^2 in $[x_{k-1}, x_k]$. Clearly since f is bounded we have $M > 0$ such that $M_k \leq M \geq m_k$ for all k. Then we have

$$\begin{aligned}U(P, f^2) - L(P, f^2) &= \sum_{k=1}^{n}(M_k^2 - m_k^2)\Delta x_k\\ &= \sum_{k=1}^{n}(M_k + m_k)(M_k - m_k)\Delta x_k\\ &\leq 2M\sum_{k=1}^{n}(M_k - m_k)\Delta x_k\\ &= 2M\{U(P, f) - L(P, f)\}\end{aligned}$$

and this establishes the integrability of f^2.

Clearly now $f \cdot g$ is integrable because we can write $f \cdot g = ((f+g)^2 - f^2 - g^2)/2$. To handle the case of f/g we note that if M_k, m_k are supremum and infimum of g in $[x_{k-1}, x_k]$ then$1/M_k, 1/m_k$ turn out to be supremum and infimum (not necessarily in corresponding order) of $1/g$ in $[x_{k-1}, x_k]$ and clearly we have

$$\begin{aligned} U(P, 1/g) - L(P, 1/g) &= \sum_{k=1}^{n} \left|\frac{1}{m_k} - \frac{1}{M_k}\right| \Delta x_k \\ &= \sum_{k=1}^{n} \left|\frac{M_k - m_k}{M_k m_k}\right| \Delta x_k \\ &< \frac{1}{m^2} \sum_{k=1}^{n} (M_k - m_k)\Delta x_k \\ &= \frac{1}{m^2} \cdot \{U(P, g) - L(P, g)\} \end{aligned}$$

so that $1/g$ is integrable on $[a, b]$ and therefore so is $f/g = f \cdot (1/g)$.

From the definition of the the Riemann integral it is obvious that it also respects the inequalities in the following manner:
If f, g, h are Riemann integrable on $[a, b]$ and if $f(x) \leq g(x) \leq h(x)$ for all $x \in [a, b]$ then we have

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx \leq \int_a^b h(x)\,dx$$

In particular we have the following two corollaries:
If f is integrable on $[a, b]$ and M, m are the supremum and infimum of f on $[a, b]$ then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

If f is integrable over $[a, b]$ then

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

Integrability on a sub-interval

We now establish another fundamental property of the Riemann integral.
If f is Riemann integrable on any closed interval then it is also integrable on any closed sub-interval.

We will provide two proofs of this statement. The first one is more natural (simpler and obvious) but limited in generality whereas the second proof is way smarter and uses Cauchy's criterion of integrability and is of a more general character.

Let f be integrable on $[a, b]$ and let $[c, d] \subseteq [a, b]$. Since f is integrable over $[a, b]$, corresponding to any $\epsilon > 0$ there exists a partition P_ϵ of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$ whenever $P \supseteq P_\epsilon$. If the points c, d are not in P_ϵ we may add them to P_ϵ without affecting the conclusion about $U(P, f) - L(P, f)$ above. Hence it is safe to assume that $c, d \in P_\epsilon$. Let $P_1 = P_\epsilon \cap [c, d]$ so that P_1 is a partition of $[c, d]$. Clearly the sum $U(P_\epsilon, f) - L(P_\epsilon, f)$ has terms coming from points of partition P_1 and since each term in the sum is non-negative, it follows that the sum

$$U(P_1, f) - L(P_1, f) \leq U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$$

Hence there is a partition P_1 of $[c, d]$ for which the Riemann's condition of integrability is satisfied. Therefore f is integrable on $[c, d]$.

Next we see how the same result can be achieved via Cauchy's condition of integrability. Since f is integrable on $[a, b]$, for any $\epsilon > 0$ we have a partition P_ϵ of $[a, b]$ such that $|S(P, f) - S(P', f)| < \epsilon$ whenever $P, P' \supseteq P_\epsilon$. If $c, d \notin P_\epsilon$ then we may add them in partition P_ϵ withou affecting the above conclusion and hence it is safe to assume that $c, d \in P_\epsilon$. Now we choose partitions P, P' in such a way that they have same points in common with intervals $[a, c]$ and $[d, b]$ but their common points with interval $[c, d]$ are arbitrary. Also while forming the sums $S(P, f), S(P', f)$ we chose the points t_k in the sub-intervals from $[a, c]$ and $[d, b]$ to be same for both partitions P, P' but those from the interval $[c, d]$ are otherwise arbitrary. In so doing we observe that all the terms coming from the sub-intervals of $[a, c]$ and $[d, b]$ cancel out in the difference $|S(P, f) - S(P', f)|$ and what remains can be expressed as $|S(P_1, f) - S(P_1', f)|$ where P_1, P_1' are arbitrary partitions of $[c, d]$ which are finer than the partition $P_0 = P_\epsilon \cap [c, d]$ of $[c, d]$. It now follows that we have

$$|S(P_1, f) - S(P_1', f)| = |S(P, f) - S(P', f)| < \epsilon$$

whenever $P_1, P_1' \supseteq P_0$. Therefore by Cauchy's criterion the function f is integrable on interval $[c, d]$.

Note that the first proof requires that each term contributing to the sum $U(P, f) - L(P, f)$ be non-negative and hence the sum corresponding to a sub-interval $[c, d]$ will never exceed the sum corresponding to the full interval $[a, b]$. But the second proof based on Cauchy's criterion does not require this non-negativity. Here the argument is very clever and powerful in the sense that from a given difference of two Riemann sums for the larger interval $[a, b]$ it generates a difference of two Riemann sums for the sub-interval $[c, d]$ such that both the differences are exactly the same, so that if the Riemann sums for the larger interval converge, then the Riemann sums for the sub-interval also converge. There are definitions of integral based on the idea of Riemann sum for which there is no corresponding concept of a Darboux upper/lower sums and in that case the first proof does not apply.

Next we have the basic result that integral is an additive interval function:
If f is integrable on $[a, b]$ and $c \in (a, b)$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

This follows from the obvious observation that a partition of $[a, b]$ can be always made finer to include $c \in (a, b)$ and hence can be regarded as a union of a partition of $[a, c]$ and a partition of $[c, b]$ and conversely any partition of $[a, c]$ can be joined with any partition of $[c, b]$ to form a partition of $[a, b]$.

Let's now define

$$\int_{a}^{a} f(x)\,dx = 0,\ \int_{b}^{a} f(x)\,dx = -\int_{a}^{b} f(x)\,dx$$

so that

$$\int_{a}^{b} f(x)\,dx + \int_{b}^{c} f(x)\,dx = \int_{a}^{c} f(x)\,dx$$

whenever f is integrable on a closed interval containing points a, b, c irrespective of the linear order of a, b, c.

After these basic properties of the Riemann integral we next focus on two important classes of functions which are Riemann integrable.

Integrability of Functions of Bounded Variation

Since a function of bounded variation can be expressed as a difference of two increasing functions it is sufficient to tackle the increasing functions. Let f be an increasing function on interval $[a, b]$. Let $x_k = a + kh$ where $h = (b - a)/n$ where positive integer n will be chosen suitably and let P be the partition made by points x_k. Clearly we have
$M_k = f(x_k), m_k = f(x_{k-1})$ so that

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{k=1}^{n}(M_k - m_k)\Delta x_k \\ &= \sum_{k=1}^{n}\{f(x_k) - f(x_{k-1})\} \cdot \frac{b-a}{n} \\ &= \frac{b-a}{n}\{f(b) - f(a)\} < \epsilon \end{aligned}$$

when $n > (b - a)\{f(b) - f(a)\}/\epsilon$. Thus Riemann's condition is satisfied for a partition P of $[a, b]$. Therefore f is integrable.

What we have established above is that:
If f is of bounded variation in $[a, b]$ then it is Riemann integrable in $[a, b]$.

Since a function of bounded variation can have a countable number of discontinuities, it follows that continuity is not necessary for integrability. However it turns out that continuity is sufficient for this purpose.

Integrability of Continuous Functions

The fact that a continuous function is integrable on a closed interval belongs to the infamous category of "results whose proofs are beyond the scope of the book/syllabus" and is unnecessarily kept in that category. As I have mentioned earlier on this blog many of the results in this category are actually much easier to prove entirely by remaining within the limits of the syllabus or the course contents, the same goes for the integrability of continuous functions.

I will present two proofs of this fact one of which is highbrow and involves the concept of uniform continuity (and therefore this result is normally left without proof in many books on calculus) and the other one which requires no higher machinery than concept of derivatives. The second proof is quite marvelous and surprising.

To keep the curiosity at a high level I will first present the proof involving uniform continuity. This is simple enough once we understand clearly that continuity on a closed interval implies uniform continuity on that interval. Hence given $\epsilon > 0$ we can find a $\delta > 0$ such that for any $x, y \in [a, b]$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon/(b - a)$. Therefore if we have a partition P of $[a, b]$ with norm $||P|| < \delta$ then clearly $M_k - m_k < \epsilon/(b - a)$ and therefore

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{k=1}^{n}\{M_k - m_k\}\Delta x_k \\ &< \frac{\epsilon}{b-a}\cdot\sum_{k=1}^{n}\Delta x_k = \epsilon \end{aligned}$$

and thus f is integrable on $[a, b]$.

The second proof requires us to consider the Darboux integrals in more detail. More formally we study the functions

$$J(x) = \overline{\int_{a}^{x}} f(t)\,dt,\ \ j(x) = \underline{\int_{a}^{x}} f(t)\,dt$$

for $x \in [a, b]$. Clearly we have by definition $J(a) = j(a) = 0$. We will show that if f is continuous at $c \in (a, b)$ then $J'(c) = j'(c) = f(c)$ and therefore if f is continuous in $[a, b]$ then both J and j have same derivatives in (a, b). This fact combined with the fact that J, j are themselves continuous in $[a, b]$ shows that $J - j$ is continuous on $[a, b]$ and its derivative vanishes in (a, b). Thus $J - j$ is constant on $[a, b]$ and hence $J(b) - j(b) = J(a) - j(a) = 0$ so that $J(b) = j(b)$ and the upper and lower Darboux integrals are equal so that f is integrable on $[a, b]$.

To complete the proof we first need to establish that J, j are continuous on $[a, b]$. Let $c \in (a, b)$ and since f is continuous at c we have a $\delta > 0$ corresponding to a given $\epsilon > 0$ such that $|f(t) - f(c)| < \epsilon$ when $|t - c| < \delta$. Let $0 < h < \delta$ then $P = \{c, c + h\}$ forms a partition of $[c, c + h]$. Let M_h, m_h be the supremum and infimum of f in $[c, c + h]$. Then clearly we have $f(c) - \epsilon \leq m_h \leq f(t) \leq M_h \leq f(c) + \epsilon$ for all $t \in [c, c + h]$. It now follows that

$$h\{f(c) - \epsilon\} \leq h\cdot m_h \leq \overline{\int_{c}^{c+h}} f(t)\,dt \leq h\cdot M_h \leq h\{f(c) + \epsilon\}$$

The upper and lower Darboux integrals follow the same rule of addition over intervals as the

Riemann integral (and the same proof applies to them for this property) hence we have

$$h\{f(c)-\epsilon\} \leq h \cdot m_h \leq \overline{\int_a^{c+h}} f(t)\,dt - \overline{\int_a^{c}} f(t)\,dt \leq h \cdot M_h \leq h\{f(c)+\epsilon\}$$

$$h\{f(c)-\epsilon\} \leq h \cdot m_h \leq J(c+h) - J(c) \leq h \cdot M_h \leq h\{f(c)+\epsilon\}$$

This shows that J is continuous from right at c. Similarly we can prove that J is continuous from left at c. Same proof can be adapted for lower Darboux integral j. The same argument can be applied for the end points of interval $[a, b]$ (the application being simpler as we have to consider only the continuity from left or from right but not from both sides). So both the functions J and j are continuous on $[a, b]$. The same argument also shows that the ratio $\{J(c+h) - J(c)\}/h$ lies between $f(c) - \epsilon$ and $f(c) + \epsilon$ for all values of h with $|h| < \delta$. This implies that $J'(c) = f(c)$. Similarly $j'(c) = f(c)$. And therefore both J, j have same derivatives in interval (a, b). This completes the proof of integrability of f without recourse to the more abstract concept of uniform continuity. This particular proof has been taken from the book *Calculus* by Michael Spivak.

Fundamental Theorems of Calculus

The above technique can be applied to show that:
If f is integrable on $[a, b]$ and a function F is defined on $[a, b]$ by the relation

$$F(x) = \int_a^x f(t)\,dt$$

then F is differentiable at $c \in [a, b]$ provided f is continuous at c and $F'(c) = f(c)$.
This result is more popularly known as the *First Fundamental Theorem of Calculus.*

What we wish to establish further is that irrespective of the fact that f is continuous or not, the function F is continuous and of bounded variation in $[a, b]$.

Clearly if $c \in [a, b]$ then

$$\begin{aligned}|F(c+h) - F(c)| &= \left|\int_a^{c+h} f(x)\,dx - \int_a^{c} f(x)\,dx\right| \\ &= \left|\int_c^{c+h} f(x)\,dx\right| \leq |c+h-h|M = |h|M\end{aligned}$$

where M is a constant such that $|f(x)| < M$ for all $x \in [a, b]$. This clearly shows that $F(c+h) - F(c)$ tends to zero with h and hence F is continuous on $[a, b]$.

On the other hand if we consider a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a, b]$ then

$$\begin{aligned}\sum_{k=1}^{n} |F(x_k) - F(x_{k-1})| &= \sum_{k=1}^{n} \left|\int_{x_{k-1}}^{x_k} f(t)\,dt\right| \\ &\leq \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f(t)|\,dt = \int_a^b |f(x)|\,dx\end{aligned}$$

and therefore F is of bounded variation in $[a, b]$.

The function F defined above is said to be an *anti-derivative* or *primitive* of f and is the main tool used for evaluating the Riemann integrals. This is established through the *Second Fundamental Theorem of Calculus:*
If f is integrable on $[a, b]$ and if there exists a function F defined on $[a, b]$ such that $F'(x) = f(x)$ for all $x \in [a, b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be a partition of $[a, b]$ and as usual let M_k, m_k be the supremum and infimum of f on interval $[x_{k-1}, x_k]$. By Lagrange's Mean Value Theorem we have

$$F(x_k) - F(x_{k-1}) = F'(t_k)(x_k - x_{k-1}) = f(t_k)(x_k - x_{k-1})$$

for some point $t_k \in (x_{k-1}, x_k)$

We clearly have $m_k \leq f(t_k) \leq M_k$ and therefore it follows that:

$$\begin{aligned} m_k(x_k - x_{k-1}) &\leq f(t_k)(x_k - x_{k-1}) \leq M_k(x_k - x_{k-1}) \\ \Rightarrow m_k(x_k - x_{k-1}) &\leq F(x_k) - F(x_{k-1}) \leq M_k(x_k - x_{k-1}) \\ \Rightarrow \sum_{k=1}^{n} m_k(x_k - x_{k-1}) &\leq \sum_{k=1}^{n} F(x_k) - F(x_{k-1}) \leq \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \\ \Rightarrow L(P, f) &\leq F(b) - F(a) \leq U(P, f) \end{aligned}$$

The above result holds for any arbitrary partition P of $[a, b]$ and since f is integrable, it clearly follows that

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

It is important to understand the difference between first and second fundamental theorems of calculus. If we assume continuity of the function being integrated the second theorem becomes a simple and obvious corollary of the first theorem. The importance of the second theorem is that it is valid even when the function being integrated is discontinuous.

In the next post we will focus on Lebesgue's (prounounced *Lebeg*) criterion for integrability.

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