Teach Yourself Limits in 8 Hours: Part 1

Introduction

While looking at certain limit problems posed in math.stackexchange.com (henceforth to be called MSE) I found that most beginners studying limits are living in a fantasy world consisting of vague notions, infinities and what not. I too had my share of such experiences during my time as a student learning calculus but I was lucky enough to get over with this phase very quickly through the help of "A Course of Pure Mathematics".

Regarding the answers posted on MSE I found that most of the answers although correct were not suitable for beginners studying limits. Some answers suggested that their authors themselves had the same vague notions but they somehow managed to avoid their pitfalls. Some other answers were using sophisticated techniques which involved deeper concepts than the concept of limits itself. And there were some heated arguments favoring one approach over another.

Therefore I decided to write a series of posts providing a step by step approach to solving limit problems encountered in an introductory calculus course. I have tried to split the whole topic into 4 posts and I believe that the gist of each post can be assimilated in not more than 2 hours and that's the logic behind the title of this series.

Contrary to my policy on this blog, I will prefer to avoid rigorous/formal proofs of the various results which I present here. This is mainly because a beginner in calculus may not be that interested in proofs and presenting these proofs at the start might serve as a detractor in learning the basic techniques. However some of the proofs will be provided in last post in the series for the sake of completeness.

Concept of a Limit

Limit is a very simple idea which can be used to study the behavior of a function when its argument takes values around a given value. Roughly speaking there are occasions when it makes sense to study the behavior of a function f defined in a neighborhood of a certain point a (but not necessarily at that point a, for example the function $f(x) = 1/x$ around point $x = 0$). For study of such situations the concept of limit was introduced. We try to figure out if the values of a function $f(x)$ tend to lie around a certain particular value when the value of x lies around a certain value. The formal definition of a limit is bit clumsy but still needs to be provided:

Limit as $x \to a$

*Let f be a function defined in a certain neighborhood of point a (**but not necessarily at a**). Then a number L is said to be the limit of $f(x)$ as x tends to a (written symbolically as $\lim\limits_{x \to a} f(x) = L$ or $f(x) \to L$ as $x \to a$) if for any given number $\epsilon > 0$ it is possible to choose a corresponding number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.*

The above definition is a bit complicated and hence some detailed explanation and remarks are necessary. First of all it should be noted that the limit of the function $f(x)$ as $x \to a$ (sometimes also called *limit of f at a*) has *has nothing to do with the value of f at a* but *has everything to do with the values of f near a*. The part $0 < |x - a| < \delta$ ensures that $x \neq a$ and also covers all values of x which are near a (the distance between x and a being less than δ). We can thus say that *a limit of a function is not necessarily a value of the function, but it is defined using the values of the function.* At the same time the limit L may be one of the values of the function f (say for example when the function $f(x) = L$ is a constant function).

The primary objective of the concept of limit is to study the behavior of a function f near some specific point a and we can loosely say that it tries to find out a pattern/trend in the values of f. The pattern sought after is to check whether all the values of the function are near some specific number L when we consider values of x near specific point a. If we can ensure that all values of f can be made to lie *as near to L as we please* for all values of x *sufficiently near* to a then only we say that the limit of f is L as $x \to a$. Thus in the definition above the inequality $|f(x) - L| < \epsilon$ serves as a *goal* (and this in reality is a tough goal which has to be achieved for every arbitrary $\epsilon > 0$) and our means to achieve the goal is to take x sufficiently close to a (or better say that we take x as close to a as is needed to achieve the goal) and this closeness is measured in terms of δ.

As one can easily observe the definition is formulated in terms of a check. It allows us to check whether a given number L is or is not the limit of a function $f(x)$ when $x \to a$. Prima facie, it does not allow us to figure out (or calculate) the limit of a function. However the above definition has turned out to be very fruitful and it allows us to derive various useful results regarding limits which can then be effectively used to calculate limit of a function. Also one should note that the *basic prerequisites for $\lim_{x \to a} f(x)$ to exist is that $f(x)$ must be defined in a certain neighborhood of a (except possibly at a).*

As a simple example we can show that

$$\lim_{x \to a} x = a$$

This is more or less obvious if we use the definition and note that here $\delta = \epsilon$ is sufficient.

Another point to note is that *if $\lim_{x \to a} f(x) = L$ exists then the function $f(x)$ is bounded in a certain neighborhood of a.* This is easy to follow if we put $\epsilon = 1$ in the definition and note that there will be a $\delta > 0$ such that $0 < |x - a| < \delta$ implies that $|f(x) - L| < 1$ i.e. $L - 1 < f(x) < L + 1$ and clearly this shows that $f(x)$ is bounded in the interval $(a - \delta, a + \delta)$.

Also note that when $x \to a$ we may have the case that x takes values greater than a. This is denoted by $x \to a+$. If x takes values less than a then we write $x \to a-$. These give rise to left-hand and right-hand limits of $f(x)$ as $x \to a$. Thus we say that $\lim\limits_{x \to a+} f(x) = L$ if for any $\epsilon > 0$ it is possible to find a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - a < \delta$. And we write $\lim\limits_{x \to a-} f(x) = L$ if for any $\epsilon > 0$ it is possible to find a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < a - x < \delta$.

Looking at the definitions of limits given above we see that $\lim_{x \to a} f(x)$ *exists if and only if both right-hand limit* $\lim_{x \to a+} f(x)$ *and and left-hand limit* $\lim_{x \to a-} f(x)$ *exist and are equal*. These concepts of left-hand and right-hand limits are useful when the definition of a function is different for values of $x < a$ and $x > a$.

Limit as $x \to \infty$ (or as $x \to -\infty$)

It is also important to discuss another limit to study the behavior of a function for large values of its argument. Then we define as follows:

Let f be a function defined on an interval of type (a, ∞) i.e. f be defined for all values of x satisfying $x > a$. Then a number L is said to be the limit of $f(x)$ as $x \to \infty$ (and written symbolically as $\lim\limits_{x \to \infty} f(x) = L$) if for any given number $\epsilon > 0$ it is possible to find a corresponding number $N > 0$ such that $|f(x) - L| < \epsilon$ whenever $x > N$.

Let f be a function defined on an interval of type $(-\infty, a)$ i.e. f be defined for all values of x satisfying $x < a$. Then a number L is said to be the limit of $f(x)$ as $x \to -\infty$ (and written symbolically as $\lim\limits_{x \to -\infty} f(x) = L$) if for any given number $\epsilon > 0$ it is possible to find a corresponding number $N < 0$ such that $|f(x) - L| < \epsilon$ whenever $x < N$.

It must be clearly understood that **the symbol ∞ used in above definitions has a meaning only in the context of these definitions and is not a number which can be operated upon via $+, -, \times, /$ (or many other operations applicable to numbers)**. Ignoring this fact is the source of all confusion prevailing in introductory calculus. In general the symbol ∞ is given a meaning via special definition such as above and its use is valid only within the context of any such definition.

We now establish the fundamental limits

$$\lim_{x \to \infty} \frac{1}{x} = 0,\ \lim_{x \to -\infty} \frac{1}{x} = 0$$

Clearly in order that the first limit is 0 we must be able to find a number $N > 0$ corresponding to any given number $\epsilon > 0$ such that $|1/x - 0| < \epsilon$ whenever $x > N$. Now we can see that if $x > N > 0$ then $|1/x - 0| = 1/x$ and this can be made smaller than ϵ if $x > 1/\epsilon$. Thus we can set $N = 1/\epsilon$ and the definition above allows us to say that $\lim_{x \to \infty} 1/x = 0$. In a similar manner we can handle $\lim_{x \to -\infty} 1/x = 0$.

Non-existence of a Limit

It is important to understand that there may be scenarios where we are not able to find any number L satisfying the definition of limit given earlier. In this case we say that the limit of a function $f(x)$ does not exist as $x \to a$(or $x \to \infty$ or $x \to -\infty$ as the case may be). We also need to study various ways (via examples) in which a function may fail to have a limit. We have the following possibilities (these are given for the case $x \to a$ but the reader may formulate the corresponding scenarios for $x \to a+, x \to a-, x \to \infty$ and $x \to -\infty$):

1. It may happen that for any given number $M > 0$ it is possible to find a number $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - a| < \delta$. In this case we write $f(x) \to \infty$ as $x \to a$ (note that this is another use of symbol ∞).
2. It may happen that for any given number $M < 0$ it is possible to find a number $\delta > 0$ such that $f(x) < M$ whenever $0 < |x - a| < \delta$. In this case we write $f(x) \to -\infty$ as $x \to a$.
3. None of the above happens. In this case we say that the function $f(x)$ *oscillates* as $x \to a$. If $f(x)$ is bounded in neighborhood of a then we say that $f(x)$ *oscillates finitely* otherwise we say that $f(x)$ *oscillates infinitely.*

We offer certain simple examples. It is not difficult to show that $1/x \to \infty$ as $x \to 0+$ and $1/x \to -\infty$ as $x \to 0-$. The examples for oscillating functions are bit tricky to understand.

Let us observe the function $f(x) = \sin(1/x)$ in the neighborhood of $x = 0$. Lets first analyze the case for $x \to 0+$. If we put $x = 2/(4n + 3)\pi$ then $f(x) = -1$ and if $x = 2/(4n + 1)\pi$ then $f(x) = 1$. Hence we can see further that given any $\delta > 0$ there exist some values of x with $0 < x < \delta$ for which $f(x) = 1$ and some other values of x say x' again satisfying $0 < x' < \delta$ for which $f(x') = -1$. Thus we don't have any number L which can satisfy the limit condition of $|f(x) - L| < \epsilon$ for all x with $0 < x - a < \delta$. Same is the case when $x \to 0-$. Since $\sin(1/x)$ is bounded the function oscillates finitely.

The same example can be modified to give a function which oscillates infinitely as $x \to 0$. Clearly we see that $1/x \to \infty$ as $x \to 0+$ hence $f(x) = (1/x)\sin(1/x)$ is unbounded and we can show (using arguments similar to those given in last paragraph) that $f(x)$ oscillates infinitely as $x \to 0$.

It is instructive to study a similar function $f(x) = x\sin(1/x)$. This function tends to limit $L = 0$ as $x \to 0$. This is because $|f(x) - L| = |x\sin(1/x)| = |x||\sin(1/x)| \leq |x|$ and hence the expression $|f(x) - L|$ can be made less than ϵ by making $|x| < \epsilon$ and thus we can choose $\delta = \epsilon$ and satisfy the definition of limit.

We don't offer a plethora of examples here as it would unnecessarily increase the length of the post. Whatever basic concepts are required have been presented and we have provided very simple examples in which a beginner won't face any difficulty. Before concluding this post we summarize the important results and examples we have developed in this post:

- For $\lim_{x\to a} f(x)$ to exist it is absolutely essential that
  - $f(x)$ is defined in a certain neighborhood of a (except possibly at a)
  - $\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x)$

- If $\lim_{x\to a} f(x)$ exists then $f(x)$ is bounded in a certain neighborhood of a
- $\displaystyle\lim_{x\to a} x = a$
- $\displaystyle\lim_{x\to\infty} \frac{1}{x} = 0,\ \lim_{x\to-\infty} \frac{1}{x} = 0$
- $1/x \to \infty$ as $x \to 0+$ and $1/x \to -\infty$ as $x \to 0-$
- Function $f(x) = \sin(1/x)$ oscillates finitely as $x \to 0$
- Function $f(x) = (1/x)\sin(1/x)$ oscillates infinitely as $x \to 0$.
- $\lim_{x\to 0} x\sin(1/x) = 0$

In the next post we will study certain limit formulas and their applications.

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