Math 105: Music & Mathematics
Test #3 Solutions

1. Starting from the ordered list 1 2 3 4 5 6 7…

a. Find the effect of applying the permutation (acg)(bf)(ed) to this list.

Answer: 7 6 1 5 4 2 3

b. Find the effect of applying (acg)(bf)(ed) again to your result from part (a).

Answer: 3 2 7 4 5 6 1

c. Use cycle notation to describe the permutation that rearranges 1 2 3 4 5 6 7 into your answer from part (b) all in one step.

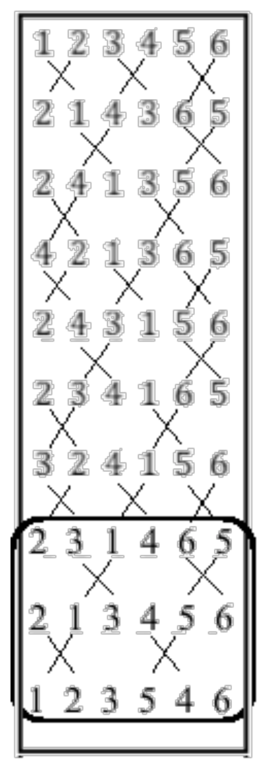
Answer: (agc)

2. A change ringing group produces changes of six bells by repeating a sequence of three permutations. The first several changes they generate using their system are shown in the diagram to the right.

a) What are the three permutations that are being used to generate these changes? Write your answers using cycle notation.

Answer: (ab)(cd)(ef), then (bc)(ef), then (ab)(ef)

b) Continue the list shown in the diagram by filling in the next three changes that will be obtained if the same pattern is continued.



Answer: Shown (circled) in the diagram to the left.

c) What permutation would rearrange the list in the first row, 1 2 3 4 5 6, into the list in the fourth row, 4 2 1 3 6 5, all in one step? Write your answer using cycle notation.

Answer: (acd)(ef)

d) If this same repeating pattern is continued long enough, how many different changes will be generated before repetition occurs? Explain your answer. (Note: do not try to answer this question by actually listing all of the changes; that's not the idea here. Find another way to make this prediction.)

Answer: 18 changes.

We can predict this because (acd)(ef) consists of a 3-cycle and a 2-cycle. When applied repeatedly, these cycles reset every third and second time, respectively, which means both reset simultaneously after $3 \times 2 = 6$ repetitions. So, we can repeat the pattern 6 times, generating 3 changes each time, for a grand total of 18 different changes.

3. For each of the following, all tones are to be selected from the set {**C, D, E, G, A, B**}. (There are six notes in this set). Show your work; make it clear how you are getting your answers. Remember that a melody is an ordered selection of notes.

a) How many ways are there to write a three note melody if no repetition of notes is allowed?

Answer: $6 \times 5 \times 4$, or 120 ways

b) How many ways are there to write a four note melody if there are no restrictions on repeating notes?

Answer: $6 \times 6 \times 6 \times 6$, or 1296 ways

c) How many ways are there to write a four note melody if repetition is allowed, with the restriction that you can't use the same note twice in a row? (For example, if the first note of the melody is an **A**, then the second note can be anything except an **A**.)

Answer: $6 \times 5 \times 5 \times 5$, or 750 ways

Note: For each note *after* the first, there are 5 options. You can use any of the original 6 notes except for the preceding note; this removes just 1 out of 6 options, leaving you with 5 each time.

4. DaNizza Pizza (at 1400 South, near campus) has a menu with sixteen different toppings. They allow you to choose three toppings for your pizza.

a. Is the selection of pizza toppings a combination, a permutation, or something else? Briefly explain your answer.

Answer: Since there is no repetition (assuming three different toppings, rather than doubling or tripling up on something), and the ordering of toppings doesn't matter (they're all mixed together on the pizza), this is best described as a **combination**.

b. How many different ways are there to choose three toppings (from the sixteen available toppings) for your pizza?

Answer: $C(16,3) = \frac{16 \times 15 \times 14}{3 \times 2 \times 1} = 8 \times 5 \times 14 = 560$ ways

c. Two of the sixteen available toppings are pepperoni and spinach. Suppose you decide that you must have pepperoni, but you absolutely don't want spinach. Subject to these restrictions, how many ways are there to choose three toppings for your pizza?

Answer: In this case you're only choosing two toppings (since pepperoni was already selected) out of fourteen options (since neither pepperoni nor spinach can be among the other two toppings). Thus, this is a combination of two toppings selected from fourteen toppings; such a selection can be made in $C(14,2) = \frac{14 \times 13}{2 \times 1} = 7 \times 13 = 91$ ways

5. Find the number of distinct rearrangements of each of the following words.

a. PERMUTE

Answer: If all 7 letters were distinct, the answer would be 7! (7 factorial), or 5040. Since there are two E's, though, we have to be more careful…

First, decide where to put the two E's. There are $C(7,2) = 21$ ways to make that choice.
After that, we can place the remaining five letters – P, R, M, U, T – in 5! = 120 ways.

So, our total is $21 \times 120 = 2520$ rearrangements

b. REARRANGE

Solution:

- Decide where to put the 3 R's; there are $C(9,3) = 84$ ways to make that choice
- Next decide where to put the 2 E's; there are $C(6,2) = 15$ ways to choose
- Next decide where to put the 2 A's; there are $C(4,2) = 6$ ways to choose
- Finally place the N,G in the remaining two spots – there are 2 ways to do this

Total: 84×15×6×2=15120 rearrangements.

6. For this problem, start by writing out the first eight rows of Pascal's triangle (Remember, the first row consists of two 1's.) Then answer the questions that appear below.

a. Write out the first eight rows of Pascal's triangle (the first two rows are provided for you):
Answer:

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

b. Circle the number in the above triangle that corresponds to $C(7,3)$. Briefly explain your choice.

Answer: The third number in the 7th row (not counting the leading 1) is 35, so $C(7,3) = 35$.
(Note: you can check this by using the combinations formula, just to make sure!)

c. What is the value of the 4th number in the 25th row of Pascal's triangle? Briefly explain how you came up with your answer. (Note: please <u>do not</u> try to write out the first 25 rows of Pascal's triangle! You don't have time for that, and it shouldn't be necessary.)

Answer: This could be interpreted in two ways, depending on whether you count the leading 1 as one of the 4 numbers. (We often skip over that leading 1, so if you did too, that's understandable).

The 25th row of Pascal's triangle starts out with

$$C(25,0) \quad C(25,1) \quad C(25,2) \quad C(25,3) \quad C(25,4) \quad \text{etc...}$$

Taken literally, the "4th number" of this row would be $C(25,3)$, whose value (using the combinations formula) is $\frac{25\times24\times23}{3\times2\times1} = 2300$.

Another answer I accepted (if you interpreted this as asking for the 4th number after the leading 1) was $C(25,4) = \frac{25\times24\times23\times22}{4\times3\times2\times1} = 12650$

7. Find the next three terms of each of the following sequences. For each, please show your work *and/or* briefly explain how you are getting your results.

a. The arithmetic sequence 3, 11, 19, ________________

Answer: 3, 11, 19, 27, 35, 43 (each term is 8 more than the preceding term)

b. The geometric sequence 2, 6, 18, ____________________

Answer: 2, 6, 18, 54, 162, 486 (each term is 3 times the preceding term)

c. The two-term recursion starting with 1, 2, and with the recursive rule $x_n = 2x_{n-1} + x_{n-2}$

1, 2, 5, 12, ________________

Answer: 1, 2, 5, 12, 29, 70, 169

(Each term is twice the preceding term plus the term before that, according to the provided recursive rule)