Teach Yourself Limits in 8 Hours: Part 3

In last two posts we have developed basic concepts and rules of limits. Continuing our journey further we now introduce certain powerful tools which help us in evaluation of limits of complicated expressions. We start with the simplest technique first.

Limits using Logarithms

In case we need to evaluate the limit of an expression of type $\{f(x)\}^{g(x)}$ then we can take logarithm and then the evaluation of limits becomes simpler. We will first illustrate the technique through an example and then provide the justification.

Let us suppose we wish to evaluate the limit

$$\lim_{x\to 0+}\left(2-e^{\arcsin^2\sqrt{x}}\right)^{3/x}$$

Let us write

$$f(x)=\left(2-e^{\arcsin^2\sqrt{x}}\right)^{3/x}$$

and then we have

$$\log f(x)=\frac{3}{x}\log\left(2-e^{\arcsin^2\sqrt{x}}\right)$$

We will first calculate the limit of $\log f(x)$ as $x\to 0+$.

$$\begin{aligned}\lim_{x\to 0}\log f(x)&=\lim_{x\to 0+}\frac{3}{x}\log\left(2-e^{\arcsin^2\sqrt{x}}\right)\\&=3\lim_{x\to 0+}\frac{\log(1+1-e^{\arcsin^2\sqrt{x}})}{1-e^{\arcsin^2\sqrt{x}}}\cdot\frac{1-e^{\arcsin^2\sqrt{x}}}{x}\\&=3\lim_{x\to 0+}1\cdot\frac{1-e^{\arcsin^2\sqrt{x}}}{x}\\&=-3\lim_{x\to 0+}\frac{e^{\arcsin^2\sqrt{x}}-1}{x}\\&=-3\lim_{x\to 0+}\frac{e^{\arcsin^2\sqrt{x}}-1}{\arcsin^2\sqrt{x}}\cdot\frac{\arcsin^2\sqrt{x}}{x}\\&=-3\lim_{x\to 0+}1\cdot\left(\frac{\arcsin\sqrt{x}}{\sqrt{x}}\right)^2\\&=-3\lim_{y\to 0+}\left(\frac{y}{\sin y}\right)^2\text{ (putting }y=\arcsin\sqrt{x})\\&=-3\cdot 1^2=-3\end{aligned}$$

Then we have $\lim_{x\to 0+}f(x)=\exp(\lim_{x\to 0+}\log f(x))=e^{-3}$. The justification of this last step is provided by the ***rule of substitution*** (provided in last post) of limits namely:

If $\lim_{x\to a}g(x)=b$ *and* $\lim_{x\to b}f(x)=L$ *and further* $g(x)\neq b$ *in a deleted neighborhood of* a*, then* $\lim_{x\to a}f\{g(x)\}=L$.

Replacing $g(x)$ by $\log g(x)$ and setting $f(x) = e^x$ in the above rule we see that $\lim_{x\to a} \log g(x) = b$ and $\lim_{x\to b} e^x = e^b$ and then $\lim_{x\to a} e^{\log g(x)} = e^b$ or

$$\lim_{x\to a} g(x) = \exp\left(\lim_{x\to a} \log g(x)\right)$$

The example limit problem is taken from MSE.

To summarize, *in order to evaluate limit of an expression of type $\{f(x)\}^{g(x)}$ we take logarithm of this expression and evaluate the limit of resulting expression. If the limit of this resulting expression is L then the limit of original expression is e^L.*

Next we study the most overused and highly powerful technique which involves concept of differentiation.

L'Hôpital's Rule

This rule is also written with simplified spelling as L'Hospital's Rule and we state its exact statement below:

Version 1: *If $f(x), g(x)$ are functions differentiable in a certain neighborhood of a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)} = L$ then we have $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = L$.*
Version 2: *If $f(x), g(x)$ are functions differentiable in a certain neighborhood of a (except possibly at a) and $|g(x)| \to \infty$ as $x \to a$ and $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)} = L$ then we have $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = L$.*

For both the versions of the rule it is also true that if $f'(x)/g'(x)$ tends to ∞ (or to $-\infty$) as $x \to a$ then so does $f(x)/g(x)$. From the statement of the rule above we can see that this rule is applicable to specially troublesome cases when substitution leads to zero numerators and zero denominators (or when the denominator tends to $\pm\infty$). In the informal / crude language we say that rule can be applied in case of indeterminate forms $0/0$ and $(\text{anything})/(\pm\infty)$. I have mentioned the word "indeterminate forms" because this is prevalent in most calculus texts although I find this term very confusing and a source of many troubles for the beginners. I prefer to state that *one should try to apply L'Hospital's rule if one is supposed to evaluate the limit of an expression of type $f(x)/g(x)$ where both numerator and denominator tend to 0 or the denominator tends to ∞ in absolute value as $x \to a$*. Moreover the rule will work only when $\lim_{x\to a} f'(x)/g'(x)$ exists (or is $\pm\infty$).

The rule is not fool-proof because under the same conditions *it may happen that $\lim_{x\to a} f(x)/g(x)$ exists but $\lim_{x\to a} f'(x)/g'(x)$ does not exist.* For example the rule fails when we try to use it to evaluate the limit

$$\lim_{x\to 0} \frac{x^2 \sin(1/x)}{x}$$

This is the case when both numerator and denominator tend to 0. And if we apply L'Hospital Rule we get the ratio

$$\frac{2x\sin(1/x)-\cos(1/x)}{1}$$

which does not tend to a limit as $x\to 0$. On the other hand the original function gets simplified to $x\sin(1/x)$ and this tends to 0 as $x\to 0$. Another example of the failure of the rule is

$$\lim_{x\to\infty}\frac{x}{x+\sin x}$$

Here both numerator and denominator tend to ∞ and the limit is easily seen to be 1. But if we apply L'Hospital's Rule we get the ratio

$$\frac{1}{1+\cos x}$$

and this does not tend to a limit as $x\to\infty$ simply because of the fact that denominator vanishes for infinitely many large values of x. The proofs of both versions of the rule will be provided in the next post.

This technique, although powerful, has some shortcomings. Sometimes differentiation can generate complicated expression and depending upon the problem multiple applications of this rule may be needed which may lead to very complicated expressions as a result of multiple differentiation. Another problem might be that differentiation itself can lead to an expression where the evaluation of limit does not seem possible. In my opinion the rule should be used only when other techniques described so far have failed. Jumping to L'Hospital's Rule for any and every limit problem is not a good idea.

I will start with the classic example where other methods don't seem to work namely

$$\lim_{x\to 0}\frac{\sin x-x}{x^3}$$

Clearly if we use L'Hospital's rule we get

$$\lim_{x\to 0}\frac{\sin x-x}{x^3}=\lim_{x\to 0}\frac{\cos x-1}{3x^2}=-\frac{1}{3}\lim_{x\to 0}\frac{1-\cos x}{x^2}=-\frac{1}{3}\cdot\frac{1}{2}=-\frac{1}{6}$$

Another example of the application of L'Hospital's rule is presented in an earlier post. Next example is from MSE:

$$\lim_{x\to 1}\frac{x}{x-1}-\frac{1}{\log x}$$

Clearly we have

$$\begin{aligned}\lim_{x\to 1}\frac{x}{x-1}-\frac{1}{\log x} &= \lim_{x\to 1}\frac{x\log x - x + 1}{(x-1)\log x}\\ &= \lim_{x\to 1}\frac{\log x}{\log x + \dfrac{x-1}{x}}\\ &= \lim_{x\to 1}\frac{1}{1+\dfrac{1}{x}\cdot\dfrac{x-1}{\log x}}\\ &= \frac{1}{1+\dfrac{1}{1}\cdot 1} = \frac{1}{2}\end{aligned}$$

In the last step we have used $\lim_{x\to 1}(x-1)/\log x = \lim_{h\to 0} h/\log(1+h) = 1$.

Next example is of a rather different kind. Suppose that $f''(a)$ exists. We will show that in this case

$$\lim_{h\to 0}\frac{f(a+h)-2f(a)+f(a-h)}{h^2} = f''(a)$$

Note that the limit on LHS may exist even though $f''(a)$ may not exist hence this should not be taken as a definition of $f''(a)$. To establish this result we first note that existence of $f''(a)$ implies the existence of $f'(x)$ in a neighbourhood of a and we can see that in this limit both numerator and denominator tend to zero as $h\to 0$. Hence we can apply L'Hospital's rule to get

$$\lim_{h\to 0}\frac{f(a+h)-2f(a)+f(a-h)}{h^2} = \lim_{h\to 0}\frac{f'(a+h)-f'(a-h)}{2h}$$

Note that we can't apply L'Hospital Rule once more as we don't know whether $f''(x)$ exists in a neighbourhood of a or not. We only have the existence of $f''(a)$. The way to proceed now is that we have to use the definition of $f''(a)$:

$$f''(a) = \lim_{h\to 0}\frac{f'(a+h)-f'(a)}{h}$$

so that

$$f'(a+h) = f'(a) + h\{f''(a)+\rho\}$$

where ρ tends to zero with h. Similarly

$$f'(a-h) = f'(a) - h\{f''(a)+\rho'\}$$

where $\rho'\to 0$ as $h\to 0$. Thus we have

$$\begin{aligned} & f'(a+h)-f'(a-h) = 2hf''(a) + h(\rho-\rho')\\ \Rightarrow\ & \frac{f'(a+h)-f'(a-h)}{2h} = f''(a) + \frac{\rho-\rho'}{2}\end{aligned}$$

Letting $h\to 0$ we get

$$\lim_{h\to 0}\frac{f'(a+h)-f'(a-h)}{2h} = f''(a)$$

A related problem is available at MSE.

Series Expansions

We finally describe the technique of using series expansions. This technique is most easily applied for problems where the limit variable tends to zero. The case $x \to a$ can be replaced by $h \to 0$ by putting $x = a + h$ so effectively the technique applies to this scenario also. The technique is based on the following result (which is more popularly known as *Taylor's Theorem*):

If $f^{(n)}(a)$ exists then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}\{f^{(n)}(a) + \rho\}$$

where ρ is an expression in a, h which tends to 0 with h.

If we put $a = 0$ and replace h by x we see that if $f^{(n)}(0)$ exists then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}\{f^{(n)}(0) + \rho\}$$

Using this formula (which can and will be proved using L'Hospital's rule in next post) we have the following series expansions:

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\{1+\rho\}\\
\sin x &= x - \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!}\{1+\rho\}\\
\cos x &= 1 - \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!}\{1+\rho\}\\
\log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n}\{1+\rho\}\\
\tan^{-1}(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n\frac{x^{2n+1}}{2n+1}\{1+\rho\}
\end{aligned}$$

In all these expressions $\rho \to 0$ as $x \to 0$. For any other specific function one may have to derive its series expansion by calculating successive derivatives. Depending upon a specific problem we decide how many terms of the expansion are needed.

Let us apply this technique to evaluate the following limit

$$\lim_{x\to 0}\frac{\tan x \tan^{-1} x - x^2}{x^6}$$

We first need to get the expansion of $\tan x$. After some labor we can see that all the even derivatives of $\tan x$ at $x = 0$ are 0 and calculation of first few odd derivatives at $x = 0$ we get

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \rho x^6$$

and we already have

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} + \rho' x^6$$

so that

$$\tan x \tan^{-1} x = x^2 + \left(\frac{2}{15} + \frac{1}{5} - \frac{1}{9}\right) x^6 + \text{ terms with higher powers of } x \text{ with } \rho, \rho'$$

It now follows that

$$\tan x \tan^{-1} x - x^2 = \frac{2}{9}x^6 + \text{ terms with higher powers of } x \text{ with } \rho, \rho'$$

and therefore we can see that

$$\lim_{x \to 0} \frac{\tan x \tan^{-1} x - x^2}{x^6} = \frac{2}{9}$$

In practice we don't write the terms containing ρ and manipulate the power series using various algebraical rules of addition, multiplication and division and assume that there will be terms containing expressions composed of ρ and higher powers of x at the end. I have shown that taking limits via the series expansion is justified because in reality the number of terms in series in finite and last term contains higher power of x and also contains expressions like ρ which tend to 0 with x.

Most calculus texts try to treat this technique in a very non-rigorous way and prefer to use infinite series and their manipulations which needs some justification using high level concepts (like uniform convergence) whereas in the above I have described a finite series expansion based on a version of Taylor's Theorem.

This technique of power series expansions should be used only when all the other techniques (rules of limits, L'Hospital's Rule) fail.

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