

Exercises in orthodox geometry

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Abstract

This collection is oriented to graduate students who want to learn fast simple tricks in geometry. (Solution of each problem requires only one non-trivial idea.)

For problem solvers. The meaning of signs next to number of the problem:

- — easy problem;
- * — hard problem;
- + — the solution requires knowledge of a theorem;
- ‡ — there are interesting solutions based on different ideas.

To get a hint, send an e-mail to the above address with the number and the name of the problem.

For problem makers. This collection is under permanent development. If you have suitable problems or corrections please e-mail it to me.

Many thanks. I want to thank everyone sharing the problems. Also I want to thank R. Matveyev, P. Petersen, S. Tabachnikov and number of students in my classes for their interest in this list and for correcting number of mistakes. I'm also thankful to everyone who took part in the discussion of this list on *mathoverflow*.

1 Curves and surfaces

1.1. Geodesic for birds. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a ℓ -Lipschitz function. Let $W \subset \mathbb{R}^3$ be epigraph of f ; i.e.

$$W = \{ (x, y, z) \in \mathbb{R}^3 \mid z \geq f(x, y) \}.$$

Show that any *geodesic* in W has *variation of turn* at most $2 \cdot \ell$.

[D. Berg],[J. Liberman]

*It is much easier to give some estimate of turn in terms of ℓ , say $\pi \cdot (1 + \ell)$.
The bound $2 \cdot \ell$ is optimal; to prove it one has to do some calculations.*

1.2. Knesev's spiral. Let γ be a plane curve with strictly monotonic curvature function. Prove that γ has no self-intersections. [Ovsienko-Tabachnikov]

1.3. *Closed curve.* A smooth closed *simple curve* with curvature at most 1 bounds a region Ω in a plane. Prove that Ω contains a disc of radius 1. ???

1.4[‡] *A curve in a sphere.* Let γ be a closed curve in a unit sphere which intersects each equator, prove that its length is at least 2π . *N. Nadirashvili*

1.5[‡] *A spring in a tin.* Let α be a closed smooth immersed curve inside a unit disc. Prove that the average absolute curvature of α is at least 1, with equality if and only if α is the unit circle possibly traversed more than once. [*S. Tabachnikov*]
If instead of a disc we have a region bounded by closed convex curve γ then it is still true that the average absolute curvature of α is at least as big as average absolute curvature of γ . The proof is not that simple, see ???.

1.6. *Shadow problem.* Let Σ be a smooth closed convex surface in Euclidean 3-space and $\gamma: [0, \ell] \rightarrow \Sigma$ be a unit speed minimizing geodesic. Set $q = \gamma(\ell)$ and $p_t = \gamma(t) - t \cdot \gamma'(t)$. Show that for any $t \in [0, \ell]$, one can not see q from the point p_t ; i.e., $\langle p_t - q, \nu_q \rangle \leq 0$, where ν_q denotes the normal vector to Σ at q . [*A. Milka, 2*]

1.7. *A minimal surface.*¹ Let Σ be a *minimal surface* in \mathbb{R}^3 which has boundary on a unit sphere. Assume Σ passes through the center of the sphere. Show that area of Σ is at least π . ???

The problem is simpler if you assume that Σ is a topological disc.

1.8. *Asymptotic line.* Consider a smooth surface Σ in \mathbb{R}^3 given as a graph $z = f(x, y)$. Let γ be a closed smooth asymptotic line on Σ . Assume Σ is *strictly saddle* in a neighborhood of γ . Prove that projection of γ on xy -plane is not star shaped. [*D. Panov, 1*]

1.9. *Closed magnetic geodesics.* Give an example of a closed Riemannian 2-manifold which has no closed smooth curve with constant geodesic curvature = 1. [*V. Ginzburg*]

1.10. *Non contractable geodesics.* Give an example of a non-flat metric on 2-torus such that it has no contractible geodesics. [*Y. Colin de Verdière, [M. Gromov, 4; 7.8(1/2)+]*]

1.11. *Fat curve.* Construct a *simple plane curve* with non-zero Lebesgue's measure. ???

1.12. *Stable net.* Let f_0 be an embedding of a graph Γ in \mathbb{S}^2 , show that there is homotopy of embeddings $f_t: \Gamma \rightarrow \mathbb{S}^2$ such that the total length of image $f_t(\Gamma)$ decreases to zero. [*Z. Brady*]

¹If Σ does not pass through the center and we only know the distance r from center to Σ then optimal bound is expected to be $\pi \cdot (1 - r^2)$. This is known if Σ is topological disc, see [Alexander–Osseman]. An analog result for area-minimizing submanifolds holds for dimensions and codimensions, see [Alexander–Hoffman–Osseman].

1.13. Oval in oval. Consider two closed smooth strictly convex planar curves, one inside another. Show that there is a chord of the outer curve, which is tangent to the inner curve at its midpoint. ???

There is a similar open problem: find a point on outer curve (not necessary convex) which has two tangent segments from this point to the inner curve has equal size.

2 Comparison geometry

For doing most of problems in this section it is enough to know second variation formula. Knowledge of some basic results such as O'Neil formula, Gauss formula, Gauss–Bonnet theorem, Toponogov's comparison theorem, Soul theorem, Toponogov splitting theorems and Synge's lemma also might help. To solve problem 2.13, it is better know that factor positively curved Riemannian manifold by an isometry group is a positively curved Alexandrov space. Problem 2.14 requires Liouville's theorem for geodesic flow. Problem 2.23 requires a Bochner type formula.

2.1. Totally geodesic hypersurface. Prove that if a compact positively curved m -manifold M admits a totally geodesic embedded hypersurface then M or its double cover is homeomorphic to the m -sphere. P. Petersen

2.2. Totally geodesic hemisphere. Let M be a simply connected m -manifold with curvature ≥ 1 ; denote by \mathbb{S}_+^2 the unit half sphere. Assume there is a totally geodesic embedding $\iota: \mathbb{S}_+^2 \hookrightarrow M$ and curvature of M is identically equal to 1 in a neighborhood of $\iota(\mathbb{S}_+^2)$. Show that M is isometric to the unit m -sphere.

2.3. Immersed convex hypersurface I. Let M be a complete simply connected Riemannian manifold with nonpositive curvature and $\dim M \geq 3$. Prove that any immersed locally convex hypersurface in M is globally convex, i.e. it is an embedded hypersurface which bounds a convex set. [S. Alexander]

2.4.* Immersed convex hypersurface II. Prove that any immersed locally convex hypersurface in a complete positively curved manifold M of dimension $m \geq 3$, is the boundary of an immersed ball. I.e. there is an immersion of a closed ball $\bar{B}^m \hookrightarrow M$ such that the induced immersion of its boundary $\partial\bar{B}^m \hookrightarrow M$ gives our hypersurface. [M. Gromov, 3], [J. Eschenburg], [B. Andrews]

2.5. Almgren's inequalities. Let Σ be a closed m -dimensional minimal surface in \mathbb{S}^n . Prove that $\text{vol } \Sigma \geq \text{vol } \mathbb{S}^m$. [F. Almgren], [M. Gromov, 1]

2.6. Hypercurve. Let $M^m \hookrightarrow \mathbb{R}^{m+2}$ be a closed smooth m -dimensional submanifold and let g be the induced Riemannian metric on M^m . Assume that sectional curvature of g is positive. Prove that curvature operator of g is positively defined. [A. Weinstein, 2]

In particular, it follows from [Micallef–Moore]/[Böhm–Wilking] that the universal cover of M is homeomorphic/diffeomorphic to a standard sphere.

2.7. Horosphere. Let M be a complete simply connected manifold with negatively pinched sectional curvature (i.e. $-a^2 \leq K \leq -b^2 < 0$). And let $\Sigma \subset M$ be an horosphere in M (i.e. Σ is a level set of a Busemann function in M). Prove that Σ with the induced intrinsic metric has *polynomial volume growth*.

V. Kapovitch

2.8. Minimal spheres. Show that a positively curved 4-manifold can not contain two distinct *equidistant minimal 2-spheres*.

D. Burago

2.9. Fixed point of conformal mappings. Let (M, g) be an even-dimensional positively curved closed oriented Riemannian manifold and $f: M \rightarrow M$ be a conformal orientation preserving map. Prove that f has a fixed point.

[A. Weinstein, 1]

2.10. Totally geodesic immersion. Let (M, g) be a simply connected positively curved m -manifold and $N \looparrowright M$ be a totally geodesic immersion. Prove that if $\dim N > \frac{m}{2}$ then N is embedded.

B. Wilking

2.11. Minimal hypersurfaces. Show that any two compact *minimal hypersurfaces* in a Riemannian manifold with positive Ricci curvature must intersect.

[T. Frankel]

2.12. Negative curvature vs. symmetry. Let (M, g) be a closed Riemannian manifold with negative Ricci curvature.

Prove that (M, g) does not admit an isometric S^1 -action.

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2.13⁺. Positive curvature and symmetry. Let (M, g) be a positively curved 4-dimensional closed Riemannian manifold with an isometric S^1 -action.

Prove that S^1 -action has at most 3 isolated fixed points.

B. Kleiner???

2.14⁺. Scalar curvature vs. injectivity radius. Let (M, g) be a closed Riemannian m -manifold with scalar curvature $\text{Sc}_g \geq m \cdot (m - 1)$ (i.e. bigger than scalar curvature of \mathbb{S}^m). Prove that the injectivity radius of (M, g) is at most π .

???

2.15. Almost flat manifold. Show that for any $\epsilon > 0$ there is $n = n(\epsilon)$ such that there is a compact n -dimensional manifold M which is not a finite factor of a *nil-manifold* and which admits a Riemannian metric with diameter ≤ 1 and sectional curvature $|K| < \epsilon$.

[G. Guzhvina]

2.16. Lie group. Show that the space of non-negatively curved left invariant metrics on a compact Lie group G is contractable.

B. Wilking

2.17. Simple geodesic. Let g be a complete Riemannian metric with positive curvature on \mathbb{R}^2 . Show that there is a two-sided infinite geodesic in (\mathbb{R}^2, g) with no self-intersections.

[V. Bangert]

2.18[‡] *Polar points.* Let (M, g) be a Riemannian manifold with sectional curvature ≥ 1 . A point $p^* \in M$ is called polar to $p \in M$ if $|px| + |xp^*| \leq \pi$ for any point $x \in M$. Prove that for any point in (M, g) there is a polar. [A. Milka, 1]

2.19. *Deformation to a product.* Let (M, g) be a compact Riemannian manifold with non-negative sectional curvature. Show that there is a continuous one parameter family of non-negatively curved metrics g_t on M , $t \in [0, 1]$, such that a finite Riemannian cover of (M, g_1) is isometric to a product of a flat torus and a simply connected manifold. [B. Wilking]

2.20* *Isometric section.* Let $s: (M, g) \rightarrow (N, h)$ be a Riemannian submersion. Assume that g is positively curved. Show that s does not admit an isometric section; i.e. there is no isometry $\iota: (N, h) \hookrightarrow (M, g)$ such that $s \circ \iota = \text{id}_N$. [G. Perelman]

2.21[‡] *Minkowski space.* Let us denote by \mathbb{M}^m the set \mathbb{R}^m equipped with the metric induced by the ℓ^p -norm. Prove that if $p \neq 2$ then \mathbb{M}^m can not be a Gromov–Hausdorff limit of Riemannian m -manifolds (M_n, g_n) such that $\text{Ric}_{g_n} \geq C$ for some constant $C \in \mathbb{R}$.

2.22. *An island of scalar curvature.* Construct a Riemannian metric g on \mathbb{R}^3 which is Euclidean outside of an open bounded set Ω and scalar curvature of g is negative in Ω . [J. Lohkamp]

2.23⁺ *If hemisphere then sphere.* Let M is an m -dimensional Riemannian manifold with Ricci curvature at least $m - 1$; moreover there is a point $p \in M$ such that sectional curvature is exactly 1 at all points on distance $\leq \frac{\pi}{2}$ from p . Show that M has constant sectional curvature. [Hang–Wang]

The problem is still interesting if instead of the first condition one has that sectional curvature ≥ 1 . If instead of first condition one only has that scalar curvature $\geq m \cdot (m - 1)$, then the conclusion does not hold; it was conjectured by Min-Oo (1995) and disproved in [Brendle–Marques–Neves]

2.24. *Local splitting.* Assume g be a Riemannian metric on \mathbb{R}^3 , such that the coordinate planes $x = 0$, $y = 0$ and $z = 0$ are flat and totally geodesic. Assume g has sectional curvature ≥ 0 (or ≤ 0) in a neighborhood of the origin. Show that origin admits a flat neighborhood.

3 Curvature free

Most of the problems in this section require no special knowledge. Solution of 3.1 relies on Gromov’s pseudo-holomorphic curves; problem 3.4 uses Liouville’s theorem for geodesic flow.

3.1⁺ *Minimal foliation.* Consider $\mathbb{S}^2 \times \mathbb{S}^2$ equipped with a Riemannian metric g which is C^∞ -close to the product metric. Prove that there is a conformally

equivalent metric $\lambda \cdot g$ and re-parametrization of $\mathbb{S}^2 \times \mathbb{S}^2$ such that each sphere $\mathbb{S}^2 \times x$ and $y \times \mathbb{S}^2$ forms a *minimal surface* in $(\mathbb{S}^2 \times \mathbb{S}^2, \lambda \cdot g)$.

3.2. Smooth doubling. Let N be a Riemannian manifold with boundary which is isometric to $(M, g)/\mathbb{S}^1$, where g is an \mathbb{S}^1 -invariant complete smooth Riemannian metric on M . Prove that the *doubling* of N is a smooth Riemannian manifold.

[A. Lytchak]

3.3. Loewner's theorem. Given $\mathbb{R}P^n$ equipped with a Riemannian metric g conformally equivalent to the canonical metric g_{can} let ℓ denote the minimal length of curves in $(\mathbb{R}P^n, g)$ not homotopic to zero. Prove that

$$\pi^n \cdot \text{vol}(\mathbb{R}P^n, g) \geq \ell^n \cdot \text{vol}(\mathbb{R}P^n, g_{\text{can}})$$

and in case of equality $g = c \cdot g_{\text{can}}$ for some positive constant c .

???

3.4⁺ Convex function vs. finite volume. Let M be a complete Riemannian manifold which admits a non-constant convex function. Prove that M has finite volume.

[S. Yau]

3.5. Besikovich inequality. Let g be a Riemannian metric on a n -dimensional cube $Q = (0, 1)^n$ such that any curve connecting opposite faces has length ≥ 1 . Prove that $\text{vol}(Q, g) \geq 1$ and equality holds if and only if (Q, g) is isometric to the interior of the unit cube.

???

3.6. Mercedes-Benz. Construct a Riemannian metric g on \mathbb{S}^3 and involution $\iota: \mathbb{S}^3 \rightarrow \mathbb{S}^3$ such that $\text{vol}(\mathbb{S}^3, g)$ is arbitrary small and $|\iota(x)|_g > 1$ for any $x \in \mathbb{S}^3$.

[C. Croke]

Note that for \mathbb{S}^2 such thing is not possible.

3.7^o Normal exponential map. Let M, N be a complete Riemannian manifolds and let N be an immersed into M . Show that the image of the *normal exponential map* of N is dense in M .

A. Lytchak

3.8. Symplectic squeezing in the torus. Let n be a positive integer. Show that arbitrary bounded domain $\Omega \subset \mathbb{C}^{n+1}$ admits an injective symplectomorphism into $\mathbb{C}^n \times \mathbb{T}$. Here we consider \mathbb{C}^n equipped with the standard symplectic structure and $\mathbb{T} = \mathbb{C}/\mathbb{Z}^2$ denotes the standard symplectic torus.

L. Polterovich, [L. Guth, 2]

Note that according to Gromov's theorem, analogous statement with $\mathbb{C}^n \times D$ as the target does not hold, here $D \subset \mathbb{C}$ is the open disc with induced symplectic structure.

4 Metric geometry.

The necessary definitions can be found in [Burago–Burago–Ivanov]. It is very hard to do 4.2 without using Kuratowski embedding. To do problem 4.5 first do problem 4.4; to do this problem you have to know a construction of compact manifolds of constant negative curvature of given dimension m . To do problem 4.12 you should be familiar with the proof of Nash–Kuiper theorem. Problems 4.13 and 4.14 are similar, in both you have to know Rademacher’s theorem on differentiability of Lipschitz maps.

4.1[◦] *Noncontracting map.* Let K be a compact metric space and $f: K \rightarrow K$ be a noncontracting map. Prove that f is an isometry.

4.2⁺ *Embedding of a compact.* Prove that any compact metric space is isometric to a subset of a compact length spaces.

4.3. *Bounded orbit.* Let X be a proper metric space and $\iota: X \rightarrow X$ is an isometry. Assume that for some $x \in X$, the orbit $\iota^n(x)$, $n \in \mathbb{Z}$ has a partial limit in X . Prove that for one (and hence for any) $y \in X$, the orbit $\iota^n(y)$ is bounded.
[A. Catka]

4.4. *Covers of figure eight.* Let (Φ, d) be a “figure eight”; i.e. a metric space which is obtained by gluing together all four ends of two unit segments.

Prove that any compact length spaces K is a Gromov–Hausdorff limit of a sequence of metric covers $(\tilde{\Phi}_n, d/n) \rightarrow (\Phi, d/n)$.
[V. Sahovic]

4.5⁺ *Constant curvature is everything.* Given integer $m \geq 2$, prove that any compact length spaces K is a Gromov–Hausdorff limit of a sequence of m -dimensional manifolds M_n with curvature $-n^2$.
[V. Sahovic]

4.6^{*} *Diameter of m -fold cover.* Let X be a length space and \tilde{X} be a connected m -fold cover of X equipped with induced intrinsic metric. Prove that

$$\text{diam } \tilde{X} \leq m \cdot \text{diam } X.$$

A. Nabutovsky, [S. Ivanov]

4.7. *2-sphere is far from a ball.* Show that there is no sequence of Riemannian metrics on \mathbb{S}^2 which converge in Gromov–Hausdorff topology to the standard ball $\bar{B}^2 \subset \mathbb{R}^2$.

4.8. *3-sphere is close to a ball.* Construct a sequence of Riemannian metrics on \mathbb{S}^3 which converge in Gromov–Hausdorff topology to the standard ball $\bar{B}^3 \subset \mathbb{R}^3$.
???

4.9[◦] *Macrodimension.* Let M be a simply connected Riemannian manifold with the following property: any closed curve can be shrunk to a point in an ε -neighborhood of itself. Prove that M is 1-dimensional on scale $10^{10} \cdot \varepsilon$; i.e. there

is a graph Γ and a continuous map $f: M \rightarrow \Gamma$, such that for any $x \in \Gamma$ we have $\text{diam}(f^{-1}(x)) \leq 10^{10} \cdot \varepsilon$. N. Zinoviev

4.10. *Anti-collapse.* Construct a sequence of Riemannian metric g_i on a 2-sphere such that $\text{vol } g_i < 1$ such that induced distance functions $d_i: \mathbb{S}^2 \times \mathbb{S}^2 \rightarrow \mathbb{R}_+$ converge to a metric $d: \mathbb{S}^2 \times \mathbb{S}^2 \rightarrow \mathbb{R}_+$ with arbitrary large Hausdorff dimension.

[Burago–Ivanov–Shoenthal]

4.11.* *No short embedding.* Construct a length-metric on \mathbb{R}^3 which admits no local *short* embeddings into \mathbb{R}^3 . [Burago–Ivanov–Shoenthal]

4.12.⁺ *Sub-Riemannian sphere.* Prove that any *sub-Riemannian metric* on the n -sphere is isometric to the intrinsic metric of a hypersurface in \mathbb{R}^{n+1} .

4.13.⁺ *Path isometry.* Show that there is no *path isometry* $\mathbb{R}^2 \rightarrow \mathbb{R}$.

4.14.⁺ *Minkowski plane.* Let \mathbb{M}^2 be a *Minkowski plane* which is not isometric to the Euclidean plane. Show that \mathbb{M}^2 does not admit a *path isometry* to \mathbb{R}^3 .

4.15. *Hyperbolic space.* Show that the hyperbolic 3-space is *quasi-isometric* to a subset of product of two hyperbolic planes. ???

4.16. *Kirszbraun's theorem.* Let $X \subset \mathbb{R}^2$ be an arbitrary subset and $f: X \rightarrow \mathbb{R}^2$ be a *short map*. Show that f can be extended as a short map to whole \mathbb{R}^2 ; i.e. there is a short map $\bar{f}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that its restriction to X coincides with f . ???

4.17. *Hilbert problem.* Let F be a convex plane figure. Construct a complete Finsler metric d on the interior of F such that any line segment in F forms a geodesic of d . ???

4.18. *Straight geodesics.* Let ρ be a length-metric on \mathbb{R}^n , which is bi-Lipschitz equivalent to the canonical metric. Assume that every *geodesic* γ in (\mathbb{R}^d, ρ) is linear (i.e., $\gamma(t) = v + w \cdot t$ for some $v, w \in \mathbb{R}^n$). Show that ρ is induced by a norm on \mathbb{R}^n . ???

4.19. *Area contracting map.* Let

$$A_\varepsilon = [0, \varepsilon] \times [0, 1] \times [0, 1] \quad \text{and} \quad B_\varepsilon = [0, \varepsilon] \times [0, \varepsilon] \times [0, \frac{1}{100 \cdot \varepsilon}].$$

Given $\varepsilon > 0$, construct a Lipschitz map $f: A_\varepsilon \rightarrow B_\varepsilon$ which is area contracting and such that the restriction $f|_{\partial A_\varepsilon}$ is a degree one map $\partial A \rightarrow \partial B_\varepsilon$.

[L. Guth, 1]

5 Topology

5.1. *Milnor's disks.* Construct two “topologically different” smooth immersions of the disk into the plane which coincide near the boundary. (Two immersions $f_1, f_2: D \hookrightarrow \mathbb{R}^2$ are topologically different if there is no diffeomorphism $h: D \rightarrow D$ such that $f_1 = f_2 \circ h$) [M. Gromov, 2; ???]

5.2. *Positive Dehn twist.* Let Σ be an oriented surface with non empty boundary. Prove that any composition of *positive Dehn twists* of Σ is not homotopic to identity *rel* boundary. R. Matveyev

5.3. *Function with no critical points.* Given $n \geq 2$, construct a smooth function f defined on a neighborhood of closed unit ball B^n in \mathbb{R}^n which has no critical points and which can not be presented in the form $\ell \circ \varphi$, where $\ell: \mathbb{R}^n \rightarrow \mathbb{R}$ is a linear function and $\varphi: B^n \rightarrow \mathbb{R}^n$ is a smooth embedding. P. Pushkar

5.4. *Conic neighborhood.* Let $p \in X$ be a point in a topological space X . We say that an open neighborhood U_p of $p \in X$ is conic if there is a homeomorphism from a cone to U_p which sends its vertex to p . Show that any two conic neighborhoods of p are homeomorphic to each other. [K. Kunen]

Note that for two cones $\text{Cone}(\Sigma_1)$ and $\text{Cone}(\Sigma_2)$ might be homeomorphic while Σ_1 and Σ_2 are not.

5.5. *Knots in C^0 -topology.* Prove that set of C^∞ -smooth embeddings $f: S^1 \rightarrow \mathbb{R}^3$ equipped with C^0 -topology is a connected space.

5.6. *Symmetric square.* Let X be a connected topological space. Note that $X \times X$ admits natural \mathbb{Z}_2 -action by $(x, y) \mapsto (y, x)$. Show that fundamental group of $X \times X / \mathbb{Z}_2$ is commutative. R. Matveyev

5.7. *Sierpinski triangle.* Find the homeomorphism group of Sierpinski triangle. B. Kliener

5.8. *Simple stabilization.* Construct two compact subsets $K_1, K_2 \subset \mathbb{R}^2$ such that K_1 is not homeomorphic to K_2 , but $K_1 \times [0, 1]$ is homeomorphic to $K_2 \times [0, 1]$. ???

5.9. *Knaster's circle.* Construct a bounded open set in \mathbb{R}^2 whose boundary does not contain a *simple curve*. [L. Wayne]

5.10. *Boundary in \mathbb{R} .* Construct three disjoint non-empty open sets in \mathbb{R} which have the same boundary.

5.11. *Deformation of homeomorphism.* Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a homeomorphism such that $|x - f(x)| < \varepsilon$ for any $x \in \mathbb{R}^n$ and some fixed $\varepsilon > 0$. Construct a

homomorphism $h: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that (1) $|x - h(x)| < \varepsilon$ for any $x \in \mathbb{R}^n$, (2) $h(x) = f(x)$ if $|x| \leq 1$ and (3) $h(x) = x$ if $|x| \geq 2$. ???

5.12° *Finite topological space.* Prove that every finite CW complex is weakly homotopy equivalent to a finite topological space, and vice versa.

6 Discrete geometry

It is suggested to do Kirszbraun's theorem (4.16) before doing problem 6.4. Before solving problem 6.7, it is better to learn what is Delone triangulation. One of the solutions of 6.10 uses mixed volumes. In order to solve problem 6.15, it is better to know what is the genus of complex curve of degree d . To solve problem 6.16 one has to use axiom of choice.

6.1° *4-polyhedron.* Give an example of a convex 4-dimensional polyhedron with 100 vertices, such that any two vertices are connected by an edge. ???

6.2. *Deformation of spherical polygon.* Let $A = a_1 a_2 \dots a_n$ and $B = b_1 b_2 \dots b_n$ be two simple spherical polygons with equal corresponding sides. Assume A lies in a hemisphere and $\angle a_i \geq \angle b_i$ for each i . Show that A is congruent to B .
[V. Zalgaller, 1]

6.3. *Pecewise linear isometry I.* Let P be a compact m -dimensional polyhedral space. Construct a pecewise linear isometry $f: P \rightarrow \mathbb{R}^m$. [V. Zalgaller, 2]

6.4⁺ *Pecewise linear isometry II.* Prove that any short map to \mathbb{R}^2 which is defined on a finite subset of \mathbb{R}^2 can be extended to a pecewise linear isometry $\mathbb{R}^2 \rightarrow \mathbb{R}^2$.
[U. Brehm]

6.5. *Minimal polyhedron.* Consider the class of polyhedral surfaces in \mathbb{R}^7 with fixed boundary curve such that each surface is homeomorphic to a 2-disc and it is glued out n triangles. Let Σ_n be a surface of minimal area in this class. Show that Σ_n is a saddle surface.

Note that it is not longer true if Σ minimizes area only in the class of polyhedral surfaces with fixed triangulation.

6.6. *Convex triangulation.* A triangulation of a convex polygon is called convex if there is a convex function which is linear on each triangle and changes the gradient if you come trough any edge of the triangulation.

Find a non-convex triangulation. [Gelfand–Kapranov–Zelevinsky]

6.7⁺ *Conic neighborhoods.* Let K be a compact length space such that for any given point $x \in K$ there is an isometry from a neighborhood of x to an open set in Euclidean cone which sends x to the vertex of cone. Prove that K is a Polyhedral space.
[N. Lebedeva]

6.8.* *A sphere with one edge.* Given a spherical polyhedral space P , denote by P_s the subset of its *singular points*.

Construct spherical polyhedral space P which is homeomorphic to \mathbb{S}^3 and such that P_s is formed by a knotted circle. Show that in such an example the total length of P_s can be arbitrary large and the angle around P_s can be made strictly less than 2π . [D. Panov, 2]

6.9. *Monotonic homotopy.* Let F be a finite set and $h_0, h_1: F \rightarrow \mathbb{R}^m$ be two maps. Consider \mathbb{R}^m as a subspace of \mathbb{R}^{2m} . Show that there is a homotopy $h_t: F \rightarrow \mathbb{R}^{2m}$ from h_0 to h_1 such that for any $x, y \in F$ the function $t \mapsto |h_t(x) - h_t(y)|$ is monotonic. [R. Alexander]

6.10[‡] *Box in a box.* Assume that one rectangular box with sizes a, b, c is inside another with sizes A, B, C . Show that $A + B + C \geq a + b + c$. [A. Shen]

6.11.* *Besicovitch's set.* Show that one can cut a unit plane disc by radii into sectors and then move each sector by a parallel translation on such a way that its union will have arbitrary small area. [A. Besicovitch]

6.12. *Boys and girls in a Lie group.* Let L_1 and L_2 be two discrete subgroups of a Lie group G , h be a left invariant metric on G and ρ_i be the induced left invariant metric on L_i . Assume $L_i \backslash G$ are compact and moreover

$$\text{vol}(L_1 \backslash (G, h)) = \text{vol}(L_2 \backslash (G, h)).$$

Prove that there is bi-Lipschitz one-to-one mapping (not necessarily a homomorphism) $f: (L_1, \rho_1) \rightarrow (L_2, \rho_2)$. D. Burago

6.13. *Telescopic group.* Given a group Γ , let us denote by $\langle \text{Tor } \Gamma \rangle$ the subgroup of Γ generated by all elements of finite order.

Construct a hyperbolic group, Γ such that for any finitely presented group G there is a subgroup $\Gamma' < \Gamma$ of finite index such that G is isomorphic to the factor group $\Gamma' / \langle \text{Tor } \Gamma' \rangle$ [Panov–Petrunin]

6.14[◦] *Round circles in \mathbb{S}^3 .* Suppose that you have a finite collection of round circles in round \mathbb{S}^3 , not necessarily all of the same radius, such that each pair is linked exactly once (in particular, no two intersect). Prove that there is an isotopy in the space of such collections of circles so that afterwards, they are all great circles. Thurston–Conway

6.15⁺ *Harnack's circles.* Prove that a smooth algebraic curve of degree d in $\mathbb{R}P^2$ consists of at most $\frac{1}{2} \cdot (d^2 - 3d + 4)$ circles. ???

6.16⁺ *Two points on each line.* Construct a set in the Euclidean plane, which intersect each line at exactly 2 points.

6.17° *Body with the same of shadows.* Two convex bodies K_1 and K_2 in Euclidean 3-space are said to have the same shadows if any shape which can appear as an orthogonal projection of K_1 can also appear as an orthogonal projection of K_2 and vice versa.

Construct two noncongruent convex bodies K_1 and K_2 which have the same shadows.

6.18° *Triangulation of a torus.* Show that torus does not admit a triangulation such that one vertex has 5 edges, one has 7 edges and all other vertexes have 6 edges. [Jendrol'–Jucovič], [Izmestiev–Kusner–Rote–Springborn]

7 Dictionary

Busemann function. Let X be a metric spaces and $\gamma: [0, \infty) \rightarrow X$ is a geodesic ray; i.e. it is a one side infinite *geodesic* which is minimizing on each interval. The Busemann function of γ is defined by

$$b_\gamma(p) = \lim_{t \rightarrow \infty} (|\gamma(t)p| - t).$$

From the triangle inequality, it is clear that the limit above is well defined.

Curvature operator. The Riemannian curvature tensor R can be viewed as an operator \mathbf{R} on bi-vectors defined by

$$\langle \mathbf{R}(X \wedge Y), Z \wedge T \rangle = \langle R(X, Y)Z, T \rangle,$$

The operator $\mathbf{R}: \wedge^2 T \rightarrow \wedge^2 T$ is called *curvature operator* and it is said to be *positively defined* if $\langle \mathbf{R}(\varphi), \varphi \rangle > 0$ for all non zero bi-vector $\varphi \in \wedge^2 T$.

Dehn twist. Let Σ be a surface and $\gamma: \mathbb{R}/\mathbb{Z} \rightarrow \Sigma$ be noncontractible closed *simple curve*. Let U_γ be a neighborhood of γ which admits a homeomorphism $h: U_\gamma \rightarrow \mathbb{R}/\mathbb{Z} \times (0, 1)$. Dehn twist along γ is a homeomorphism $f: \Sigma \rightarrow \Sigma$ which is identity outside of U_γ and $h \circ f \circ h^{-1}: (x, y) \mapsto (x+y, y)$. If Σ is orientable, then the Dehn twist described above is called *positive* if h is orientation preserving.

Doubling of a manifold M with boundary ∂M is two copies of M_1, M_2 identified along corresponding points on the boundary $\partial M_1, \partial M_2$.

Euclidean cone. Let Σ be a metric space with diameter $\leq \pi$. A metric space K is called Euclidean cone over Σ if its underling set coincides with the quotient $\Sigma \times [0, \infty)/\sim$, where $(x, 0) \sim (y, 0)$ for any $x, y \in \Sigma$ and the metric is defined by cosine rule; i.e.

$$|(x, a)(y, b)|_K^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos |xy|_\Sigma.$$

Equidistant subsets. Two subsets A and B in a metric space are called equidistant if dist_A is constant on B and dist_B is constant on A .

Geodesic. Let X be a metric space and \mathbb{I} be a real interval. A locally isometric immersion $\gamma: \mathbb{I} \rightarrow X$ is called geodesic. In other words, γ is a geodesic if for any $t_0 \in \mathbb{I}$ we have $|\gamma(t)\gamma(t')| = |t - t'|$ for all $t, t' \in \mathbb{I}$ sufficiently close to t_0 . Note that in our definition geodesic has unit speed (that is not quite standard).

Length space. A complete metric space X is called *length space* if the distance between any pair of points in X is equal to the infimum of lengths of curves connecting these points.

Minimal surface. Let Σ be a k -dimensional smooth surface in a Riemannian manifold M and $T = T\Sigma$ and $N = N\Sigma$ correspondingly tangent and normal bundle. Let $s: T \otimes T \rightarrow N$ denotes the second fundamental form of Σ . Let e_i is an orthonormal basis for T_x , set $H_x = \sum_i s(e_i, e_i) \in N_x$; it is the mean curvature vector at $x \in \Sigma$. We say that Σ is *minimal* if $H \equiv 0$.

Minkowski space — \mathbb{R}^m with a metric induced by a norm.

Nil-manifolds form the minimal class of manifolds which includes a point, and has the following property: the total space of any oriented S^1 -bundle over a nil-manifold is a nil-manifold.

It also can be defined as a factor of a connected nilpotent Lie group by a lattice.

Normal exponential map ???

Path isometry A map $f: X \rightarrow Y$ of *length spaces* X and Y is a path isometry if for any path $\alpha: [0, 1] \rightarrow X$, we have

$$\text{length}(\alpha) = \text{length}(f \circ \alpha).$$

Polyhedral space — a simplicial complex with a metric such that each simplex is isometric to a simplex in a Euclidean space.

It admits the following generalizations: *spherical (hyperbolic)* polyhedral space — a simplicial complex with a metric such that each simplex is isometric to a simplex in a unit sphere (corresp. hyperbolic space of constant curvature -1).

A point in a polyhedral space is called *regular* if it has a neighborhood isometric to an open set in a Euclidean space; otherwise it called *singular*. To define regular/singular points in spherical or hyperbolic space, one has to exchange in the above definition Euclidean space to unit sphere or hyperbolic space with curvature -1 .

Polynomial volume growth. A Riemannian manifold M has polynomial volume growth if for some (and therefore any) $p \in M$, we have $\text{vol } B_r(p) \leq C \cdot (r^k + 1)$, where $B_r(p)$ is the ball in M and C, k are real constants.

Proper metric space. A metric space X is called *proper* if any closed bounded set in X is compact.

Piecewise linear isometry — a piecewise linear map from a polyhedral space which is isometric on each simplex. More precisely: Let P and Q be polyhedral spaces, a map $f: P \rightarrow Q$ is called piecewise linear isometry if there is a triangulation \mathcal{T} of P such that at any simplex $\Delta \in \mathcal{T}$ the restriction $f|_{\Delta}$ is globally isometric.

Quasi-isometry. A map $f: X \rightarrow Y$ is called a quasi-isometry if there is a constant $C < \infty$ such that $f(X)$ is a C -net in Y and

$$\frac{|xy|}{C} - C \leq |f(x)f(y)| \leq C \cdot |xy| + C.$$

Note that a quasi-isometry is not assumed to be continuous, for example any map between compact metric spaces is a quasi-isometry.

Saddle surface. A smooth surface Σ in \mathbb{R}^3 is saddle (correspondingly strictly saddle) if the product of the principle curvatures at each point is ≤ 0 (correspondingly < 0).

It admits the following generalization to non-smooth case and arbitrary dimension of the ambient space: A surface Σ in \mathbb{R}^n is saddle if the restriction $\ell|_{\Sigma}$ of any linear function $\ell: \mathbb{R}^3 \rightarrow \mathbb{R}$ has no strict local minima at interior points of Σ .

One can generalize it further to an arbitrary ambient space, using convex functions instead of linear functions in the above definition.

Short map — a distance non increasing map.

Simple curve — an image of a continuous injective map of a real segment or a circle.

Sub-Riemannian metric.

Variation of turn. Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ be a curve. The variation of turn of γ is defined as supremum of sum of ??? angles for broken lines inscribed in γ . Namely,

$$\sup \left\{ \sum_{i=1}^{n-1} \alpha_i \mid a = t_0 < t_1 < \dots < t_n = b \right\},$$

where $\alpha_i = \pi - \angle \gamma(t_{i-1})\gamma(t_i)\gamma(t_{i+1})$.

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