

On intrinsic isometries to Euclidean space.

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Abstract

I consider compact metric spaces which admit intrinsic isometries to Euclidean d -space. The main result roughly states that the class of these spaces coincides with class of inverse limits of Euclidean d -polyhedra.

1 Introduction

The *intrinsic isometries* are defined in section 2; it is a variation of notion of *path isometry*, i.e. map which preserves the lengths of curves. Any intrinsic isometry is a path isometry, the converse does not hold in general.

The following statement is one of the reason we prefer intrinsic isometry.

1.1. Trivial statement. *Let \mathcal{X} be a compact metric space which admits an intrinsic isometry to d -dimensional Euclidean space (further denoted by \mathbb{E}^d). Then $\dim \mathcal{X} \leq d$, where \dim denotes the Lebesgue's covering dimension.*

This statement is proved in section 3. An analogous statement for path isometry does not hold, see example 4.2. The Hausdorff dimension can not be bounded on the similar way. For example, \mathbb{R} -tree admits an intrinsic isometry to \mathbb{R} and it contains compact subsets of arbitrary large Hausdorff dimension.

Here are some known results on length spaces which admit *intrinsic isometry* to \mathbb{E}^d .

1.2. Theorem. *Let \mathcal{R} be d -dimensional Riemannian space and $f: \mathcal{R} \rightarrow \mathbb{E}^d$ be a short map¹. Then given $\varepsilon > 0$, there is an intrinsic isometry $\iota: \mathcal{R} \rightarrow \mathbb{E}^d$ such that*

$$|f(x)\iota(x)|_{\mathbb{E}^d} < \varepsilon$$

for any $x \in \mathcal{R}$.

In particular, any Riemannian d -space admits an intrinsic isometry to \mathbb{E}^d .

For path isometries, this theorem was proved in [Gromov, 2.4.11], and the same proof works for intrinsic isometries. Applying this theorem, one can show that any limit of increasing sequence of Riemannian metrics on a fixed d -dimensional manifold admits an intrinsic isometry to \mathbb{E}^d . (The proof is similar to “if”-part of the main theorem.) In particular, any sub-Riemannian metric on d -dimensional manifold admits an intrinsic isometry to \mathbb{E}^d .

1.3. Theorem. *Let \mathcal{P} be a Euclidean polyhedron and $f: \mathcal{P} \rightarrow \mathbb{E}^d$ be a short map. Then, given $\varepsilon > 0$, there is a piecewise linear intrinsic isometry $\iota: \mathcal{P} \rightarrow \mathbb{E}^d$ such that*

$$|f(x)\iota(x)|_{\mathbb{E}^d} < \varepsilon$$

¹i.e. 1-Lipschitz map

for any $x \in \mathcal{P}$.

1.4. Corollary. *Any d -dimensional Euclidean polyhedron admits a piecewise linear intrinsic isometry to \mathbb{E}^d .*

The corollary was proved in [Zalgaller] for dimension ≤ 4 , but a slight modification of the proof works in all dimensions, see [Krat]. The 2-dimensional case of this theorem was proved in [Krat]. Later, the proof was extended to all dimensions in [Akopyan]; the proof uses a *piecewise linear analog of Nash–Kuiper theorem*. This theorem was proved in [Brehm], but this work was left without attention for many years and reproved independently in [Akopyan–Tarasov].

Iff-condition. Now we describe the main result of the paper.

A compact metric space \mathcal{X} is called *pro-Euclidean space of rank $\leq d$* if it can be presented as an *inverse limit* $\mathcal{X} = \varprojlim \mathcal{P}_n$ (see section 2) of a sequence of Euclidean d -polyhedra \mathcal{P}_n .

1.5. Main theorem. *A compact metric space \mathcal{X} admits an intrinsic isometry to \mathbb{E}^d if and only if \mathcal{X} is a pro-Euclidean space of rank $\leq d$.*

The proof is straightforward. I like the formulation of the theorem; it seems to be the first case when inverse limits help to solve a natural problem in metric geometry.

Note that the statement in theorem 1.2 (in the compact case) is equivalent to the fact that any compact Riemannian d -space is a pro-Euclidean space of rank $\leq d$. The latter can be obtained directly from the following exercise; this way the main theorem provides an alternative proof to theorem 1.2 in the compact case.

1.6. Exercise. *Show that any compact Riemannian space admits a Lipschitz approximation by Euclidean polyhedra.*

A non-example. Let us remind that *Minkowski space* is finite dimensional real vector spaces with metric induced by a norm.

1.7. Proposition. *Let Ω be an open subset of Minkowski d -space \mathbb{M}^d . Assume Ω admits an intrinsic² isometry to \mathbb{E}^m then $d \leq m$ and \mathbb{M}^d is isometric to a \mathbb{E}^d .*

In particular, the condition 1.1 on Lebesgue’s dimension is not sufficient.

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2 Preliminaries

Standard definitions. Given a metric space \mathcal{X} and two points $x, x' \in \mathcal{X}$, we will denote by $|xx'| = |xx'|_{\mathcal{X}}$ the distance from x to x' in \mathcal{X} .

A *length space* is a metric space such that for any two points x, x' the distance $|xx'|$ coincides with the infimum of lengths of curves connecting x and x' .

²In fact the same is true for path isometry.

A map $f: \mathcal{X} \rightarrow \mathcal{Y}$ between metric spaces \mathcal{X} and \mathcal{Y} is called *short* if for any $x, x' \in \mathcal{X}$ we have

$$|f(x)f(x')|_{\mathcal{Y}} \leq |xx'|_{\mathcal{X}}.$$

A length space \mathcal{P} is called *Euclidean d -polyhedron* if there is a finite triangulation of \mathcal{P} such that each simplex is isometric to a simplex in \mathbb{E}^d .

Inverse limit. Consider an *inverse system* of compact metric spaces $(\mathcal{X}_n)_{n=0}^{\infty}$ and short maps $\varphi_{m,n}: \mathcal{X}_m \rightarrow \mathcal{X}_n$ for $m \geq n$; i.e.,

1. $\varphi_{m,n} \circ \varphi_{k,m} = \varphi_{k,n}$ for any triple $k \geq m \geq n$ and
2. for any n , the map $\varphi_{n,n}$ is identity map of \mathcal{X}_n .

A compact metric space \mathcal{X} is called *inverse limit of the system* $(\varphi_{m,n}, \mathcal{X}_n)$ (denoted by $\mathcal{X} = \varprojlim \mathcal{X}_n$) if its underlying space consists of all sequences $x_n \in \mathcal{X}_n$ such that $\varphi_{m,n}(x_m) = x_n$ for all $m \geq n$ and for two such sequences (x_n) and (x'_n) the distance is defined by

$$|(x_n)(x'_n)|_{\mathcal{X}} = \lim_{n \rightarrow \infty} |x_n x'_n|_{\mathcal{X}_n}.$$

If $\mathcal{X} = \varprojlim \mathcal{X}_n$, then the map $\psi_n: \mathcal{X} \rightarrow \mathcal{X}_n$, defined by $\psi_n: (x_i)_{i=0}^{\infty} \mapsto x_n$ are called *projections*. Clearly $\psi_n = \varphi_{m,n} \circ \psi_m$ for all $m \geq n$.

Comments. The above definition is equivalent to the usual inverse limit in the category with class of objects formed by compact metric spaces and class of morphisms formed by short maps.

Note that inverse limit is not always defined, and if defined it is the result is compact by definition. (In principle, the category of compact metric spaces can be extended so that the limit of any inverse system is well defined.)

It is easy to see that inverse limit of length spaces is a length space.

In general, the inverse limit of a system of spaces differ from its Gromov–Hausdorff limit. For example, consider inverse system $\mathcal{X}_n = [0, 1]$ with maps $\varphi_{m,n}(x) \equiv 0$. The inverse limit of this system is isometric to one-point space, while the Gromov–Hausdorff limit is isometric to $[0, 1]$. Nevertheless, it is easy to see that if for any $\varepsilon > 0$ the images of $\varphi_{m,n}$ form an ε -net in \mathcal{X}_n for all sufficiently large m and n , then $\mathcal{X} = \varprojlim \mathcal{X}_n$ is isometric to the Gromov–Hausdorff limit.

Intrinsic isometries and pull back metrics. Let \mathcal{X} and \mathcal{Y} be metric spaces and $f: \mathcal{X} \rightarrow \mathcal{Y}$ be continuous map. Given two points $x, x' \in \mathcal{X}$, a sequence of points $x = x_0, x_1, \dots, x_n = x'$ is called ε -chain from x to x' if $|x_{i-1}x_i| \leq \varepsilon$ for all $i > 0$. Set

$$\text{pull}_{f,\varepsilon}(x, x') = \inf \left\{ \sum_{i=1}^n |f(x_{i-1})f(x_i)|_{\mathcal{Y}} \right\}$$

where the infimum is taken along all ε -chains $(x_i)_{i=0}^n$ from x to x' .

Clearly $\text{pull}_{f,\varepsilon}$ is a pre-metric³ on \mathcal{X} and $\text{pull}_{f,\varepsilon}(x, x')$ is non-increasing in ε . Thus, the following (possibly infinite) limit

$$\text{pull}_f(x, x') = \lim_{\varepsilon \rightarrow 0} \text{pull}_{f,\varepsilon}(x, x')$$

³i.e. it satisfies triangle inequality, it is symmetric, non-negative and $\text{pull}_{f,\varepsilon}(x, x) = 0$, but it might happen that $\text{pull}_{f,\varepsilon}(x, x') = 0$ for $x \neq x'$.

is well defined. The pre-metric $\text{pull}_f: \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty]$ will be called *pull back metric* for f .

A map $f: \mathcal{X} \rightarrow \mathcal{Y}$ between length spaces \mathcal{X} and \mathcal{Y} is an *intrinsic isometry* if

$$|xx'|_{\mathcal{X}} = \text{pull}_f(x, x')$$

for any $x, x' \in \mathcal{X}$.

Any intrinsic isometry is a short map. Moreover, it is easy to see that intrinsic isometry preserves the lengths of curves. The converse does not hold, see section 4.

2.1. Proposition. *Let \mathcal{X} be a compact (or even proper⁴) metric space. Then existence of intrinsic isometry $f: \mathcal{X} \rightarrow \mathcal{Y}$ implies that \mathcal{X} is a length space.*

The proof is left to the reader. Note that for general \mathcal{X} it is not true. Consider two points which connected by countable number of unit intervals \mathbb{I}_n and one interval of length $\frac{1}{2}$; equip the obtained space with natural intrinsic metric. Let us remove from our space the interval of length $\frac{1}{2}$. The metric on the remaining space \mathcal{X} is not intrinsic. Further let us construct a map $f: \mathcal{X} \rightarrow \mathbb{R}$ so that the restriction $f_n = f|_{\mathbb{I}_n}$ is an intrinsic isometry, $f_n(0) = 0$, $f_n(1) = \frac{1}{2}$ and $f_n(x)$ converges uniformly to $\frac{x}{2}$. It is easy to see that $f: \mathcal{X} \rightarrow \mathbb{R}$ is an intrinsic isometry.

2.2. Proposition. *Let \mathcal{X} and \mathcal{Y} be metric spaces, \mathcal{X} be compact and the continuous map $f: \mathcal{X} \rightarrow \mathcal{Y}$ is such that*

$$\sup_{x, x' \in \mathcal{X}} \text{pull}_f(x, x') < \infty.$$

Then given $\varepsilon > 0$ there is $\delta = \delta(f, \varepsilon) > 0$ such that for any short map $h: \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$|f(x)h(x)|_{\mathcal{Y}} < \delta \text{ for any } x \in \mathcal{X}$$

we have

$$\text{pull}_f(x, x') < \text{pull}_h(x, x') + \varepsilon$$

for any $x, x' \in \mathcal{X}$.

The proof is a direct application of the lemma 2.3.

For a compact metric space \mathcal{X} , we denote by $\text{pack}_\varepsilon \mathcal{X}$ the maximal number of points in \mathcal{X} on distance $> \varepsilon$ from each other. Clearly $\text{pack}_\varepsilon \mathcal{X}$ is finite for any $\varepsilon > 0$.

2.3. Lemma. *Let \mathcal{X} and \mathcal{Y} be metric spaces, \mathcal{X} is compact and $f, h: \mathcal{X} \rightarrow \mathcal{Y}$ be two continuous maps.*

Assume for any $x \in \mathcal{X}$, $|f(x)h(x)| < \delta$ then for any $x, x' \in \mathcal{X}$ we have

$$\text{pull}_{f, \varepsilon}(x, x') \leq \text{pull}_{h, \varepsilon}(x, x') + 4 \cdot \delta \cdot \text{pack}_\varepsilon \mathcal{X}.$$

Proof. Assume $\text{pull}_{h, \varepsilon}(x, x') < \ell$, i.e. there is an ε -chain $\{x_i\}_{i=0}^n$ from x to x' such that

$$\sum_{i=1}^n |h(x_{i-1})h(x_i)|_{\mathcal{Y}} < \ell. \quad (*)$$

⁴I.e. all closed bounded sets in \mathcal{X} are compact.

Since $|h(x_i)f(x_i)| < \delta$,

$$\text{pull}_{f,\varepsilon}(x, x') \leq \sum_{i=1}^n |f(x_{i-1})f(x_i)|_{\mathcal{Y}} < \sum_{i=1}^n |h(x_{i-1})h(x_i)|_{\mathcal{Y}} + 2 \cdot n \cdot \delta$$

Assume n is the smallest number for which there is an ε -chain satisfying (*). It is enough to show that

$$n < 2 \cdot \text{pack}_{\varepsilon} \mathcal{X}.$$

If $n \geq 2 \cdot \text{pack}_{\varepsilon} \mathcal{X}$, there are i and j such that $j - i > 1$ and $|x_i x_j| \leq \varepsilon$. Remove from this chain all elements x_k with $i < k < j$; i.e. consider new ε -chain

$$x = x_0, \dots, x_{i-1}, x_i, x_j, x_{j+1}, \dots, x_n = x'$$

By triangle inequality in \mathcal{Y} , the new chain satisfies (*); i.e. n is not the smallest number, a contradiction. \square

2.4. Proposition. *Let \mathcal{X} and \mathcal{Y} be metric spaces, \mathcal{X} be compact and $\iota: \mathcal{X} \rightarrow \mathcal{Y}$ be an intrinsic isometry.*

Then given $\varepsilon > 0$ there is $\delta = \delta(\iota, \varepsilon) > 0$ such that for any connected set $W \subset \mathcal{X}$

$$\text{diam } \iota(W) < \delta \implies \text{diam } W < \varepsilon.$$

Proof. Assume contrary, i.e. there is a sequence of connected subsets $W_n \subset \mathcal{X}$ such that $\text{diam } \iota(W_n) \rightarrow 0$ as $n \rightarrow \infty$ but $\text{diam } W_n > \varepsilon$. Thus there are two sequences of points $x_n, x'_n \in W_n$ such that $|x_n x'_n| \geq \varepsilon$. Pass to a subsequence of n so that $W_n \rightarrow W$ in Hausdorff sense and $x_n \rightarrow x, x'_n \rightarrow x'$. We obtain a closed connected subset $W \subset \mathcal{X}$ with two distinct points x and x' such that $\iota(W) = p$ for some $p \in \mathcal{Y}$.

Since W is connected, for any $\varepsilon > 0$ there is an ε -chain $(x_i)_{i=0}^n$ from x to x' such that $\iota(x_i) = p$ for all i . Thus, we have $\text{pull}_{\iota, \varepsilon}(x, x') = 0$ for any $\varepsilon > 0$; i.e., $\text{pull}_{\iota}(x, x') = 0$, a contradiction. \square

3 The proofs

Proof of the trivial statement (1.1). Given $\varepsilon > 0$ choose $\delta = \delta(\iota, \varepsilon)$ as in proposition 2.4. Since $\dim \mathbb{E}^d = d$, there is a finite open covering $\{U_i\}_{i=1}^n$ of $\iota(\mathcal{X})$ with multiplicity $\leq d + 1$ and such that $\text{diam } U_i < \delta$ for each i .

Consider the covering $\{V_{\alpha}\}$ of \mathcal{X} by connected components of $\iota^{-1}(U_i)$ for all i . According to proposition 2.1, \mathcal{X} is a length space. In particular, all sets V_{α} are open. According to proposition 2.4, $\text{diam } V_{\alpha} < \varepsilon$. Clearly multiplicity of $\{V_{\alpha}\}$ is at most $d + 1$. Thus, the statement follows. \square

Proof of "if" in the main theorem (1.5). Let \mathcal{X} be a pro-Euclidean space of rank $\leq d$. Assume $(\mathcal{P}_n)_{n=0}^{\infty}$ is a sequence of d -dimensional Euclidean polyhedra and

$$\varphi_{m,n}: \mathcal{P}_m \rightarrow \mathcal{P}_n$$

is an inverse system of short maps such that $\mathcal{X} = \varprojlim \mathcal{P}_n$. Let $\psi_n: \mathcal{X} \rightarrow \mathcal{P}_n$ be the projections.

According to theorem 1.3, given $\varepsilon_{n+1} > 0$ and a piecewise linear intrinsic isometry $\iota_n: \mathcal{P}_n \rightarrow \mathbb{E}^d$ there is a piecewise linear intrinsic isometry $\iota_{n+1}: \mathcal{P}_{n+1} \rightarrow \mathbb{E}^d$ such that

$$|\iota_{n+1}(x) \iota_n \circ \varphi_{n+1,n}(x)| < \varepsilon_{n+1}.$$

for any $x \in \mathcal{P}_n$. It remains to show that sequence ε_n can be chosen on such a way that $\iota_n \circ \psi_n$ converges to an intrinsic isometry $\iota: \mathcal{X} \rightarrow \mathbb{E}^d$.

Let us choose $\varepsilon_{n+1} > 0$ so that

$$\varepsilon_{n+1} < \frac{1}{2} \min\{\varepsilon_n, \delta(\iota_n, \frac{1}{n})\},$$

where $\delta(\iota_n, \frac{1}{n})$ as in proposition 2.2. Clearly, $\sum_i \varepsilon_i < \infty$, thus the the following limit exists

$$\iota = \lim_{n \rightarrow \infty} \iota_n \circ \psi_n, \quad \iota: \mathcal{X} \rightarrow \mathbb{E}^d.$$

Obviously, ι is short. Further, for any $x \in \mathcal{X}$,

$$|\iota(x) \iota_n \circ \psi_n(x)| < \sum_{i=n+1}^{\infty} \varepsilon_i < \delta(\iota_n, \frac{1}{n}).$$

Thus, according to proposition 2.2,

$$\text{pull}_{\iota}(x, x') + \frac{1}{n} > \text{pull}_{\iota_n \circ \psi_n}(x, x') \geq |\psi_n(x) \psi_n(x')|_{\mathcal{P}_n}.$$

Since $|\psi_n(x) \psi_n(x')|_{\mathcal{P}_n} \rightarrow |xx'|_{\mathcal{X}}$ as $n \rightarrow \infty$, the map $\iota: \mathcal{X} \rightarrow \mathbb{E}^d$ is an intrinsic isometry. \square

Proof of “only if” in the main theorem (1.5). We will give a construction a polyhedron \mathcal{P} associated to an intrinsic isometry $\iota: \mathcal{X} \rightarrow \mathbb{E}^d$ and a tiling of \mathbb{E}^d by coordinate a -cubes. (The space \mathcal{P} will be glued out of a -cubes.) The construction will be done in such a way that if a tiling τ' is a subdivision of a tiling τ then for corresponding polyhedra \mathcal{P}' and \mathcal{P} there will be a natural intrinsic isometry $\mathcal{P}' \rightarrow \mathcal{P}$. Thus we will construct the needed inverse system of polyhedra out of nested subdivisions of \mathbb{E}^d .

Take sequences $a_n = \frac{1}{2^n}$ and set $r_n = \frac{1}{10} \cdot a_n$. Fix n for a while and consider tiling of \mathbb{E}^d by coordinate a_n -cubes. Let us construct a Euclidean polyhedron \mathcal{P}_n associated to this tiling.

The image $\iota(\mathcal{X})$ is covered by finite number of such a_n -cubes, say $\{\square_n^i\}$. For each \square_n^i , consider all connected components $\{W_n^{ij}\}$ of

$$B_{r_n}(\iota^{-1}(\square_n^i)) \subset \mathcal{X},$$

where $B_r(S)$ denotes r -neighborhood of set S .

According to proposition 2.1, \mathcal{X} is a length space. In particular, each set W_n^{ij} is open and contains a ball of radius r_n . Thus for fixed i the collection of open sets $\{W_n^{ij}\}$ is finite. Therefore the set of all $\{W_n^{ij}\}$ for all $\{\square_n^i\}$ forms a finite open cover of \mathcal{X} . For each W_n^{ij} make an isometric copy \square_n^{ij} of \square_n^i and fix an isometry $\iota_n^{ij}: \square_n^{ij} \rightarrow \square_n^i$. The Euclidean polyhedron \mathcal{P}_n , is glued from \square_n^{ij} by the following rule: glue $\square_n^{i_1 j_1}$ to $\square_n^{i_2 j_2}$ along $(\iota_n^{i_2 j_2})^{-1} \circ \iota_n^{i_1 j_1}$ iff $W_n^{i_1 j_1} \cap W_n^{i_2 j_2} \neq \emptyset$. (The map $(\iota_n^{i_2 j_2})^{-1} \circ \iota_n^{i_1 j_1}$ sends one of the faces of $\square_n^{i_1 j_1}$ isometrically to a face of $\square_n^{i_2 j_2}$.)

The constructed polyhedron \mathcal{P}_n admits a natural piecewise linear intrinsic isometry $\iota_n: \mathcal{P}_n \rightarrow \mathbb{E}^d$, defined as $\iota_n(x) = \iota_n^{ij}(x)$ if $x \in \square_n^{ij}$. Further, there is uniquely defined intrinsic isometry $\varphi_{m,n}: \mathcal{P}_m \rightarrow \mathcal{P}_n$ for $m \geq n$ which satisfies $\iota_m = \iota_n \circ \varphi_{m,n}$ and

$$\varphi_{m,n}(\square_m^{i'j'}) \subset \square_n^{ij} \subset \mathcal{P}_n \Rightarrow W_m^{i'j'} \subset W_n^{ij} \subset \mathcal{X}.$$

Further, set $\psi_n: \mathcal{X} \rightarrow \mathcal{P}_n$ to be intrinsic isometry which uniquely determined by $\iota_n \circ \psi_n = \iota$ and

$$\psi_n(x) \in \square_n^{ij} \subset \mathcal{P}_n \Rightarrow x \in W_n^{ij} \subset \mathcal{X}.$$

Clearly, \mathcal{P}_n together with $\varphi_{m,n}$ form an inverse system and $\psi_n = \varphi_{m,n} \circ \psi_m$ for any pair $m \geq n$.

In order to prove that $\mathcal{X} = \varprojlim \mathcal{P}_n$, it only remains to show that

$$|xx'|_{\mathcal{X}} \leq \lim_{n \rightarrow \infty} |\psi_n(x) \psi_n(x')| \quad (*)$$

for all $x, x' \in \mathcal{X}$.

Given a subset $K \subset \mathcal{P}_n$, let us denote by $K^* \subset \mathcal{X}$ the union of all $W_n^{ij} \subset \mathcal{X}$ such that $\square_n^{ij} \cap K \neq \emptyset$. Clearly, if K is connected then so is K^* . More over, $\iota(K^*) \subset B_{r_n}(\iota_n(K))$. Thus, from proposition 2.4, we have that for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$r_n + \text{diam } K < \delta \implies \text{diam } K^* < \varepsilon \quad (**)$$

Assume (*) is wrong, then one can choose $x, x' \in \mathcal{X}$ and $\varepsilon, \ell > 0$ so that

$$\text{pull}_{i,\varepsilon}(x, x') > \ell > |\psi_n(x) \psi_n(x')|_{\mathcal{P}_n} \quad (**)$$

for all n . In particular, for all n there is a path $\gamma_n: [0, 1] \rightarrow \mathcal{P}_n$ from $\psi_n(x)$ to $\psi_n(x')$ with length $< \ell$. Choose $\delta = \delta(\varepsilon, \ell)$ as in proposition 2.4. Let $0 = t_0 < t_1 < \dots < t_m = 1$ be such that

$$\text{diam } \gamma([t_{i-1}, t_i]) < \frac{\delta}{2}. \quad (**)$$

Clearly one can assume that $m \leq 2 \cdot \lceil \frac{\ell}{\delta} \rceil$. For each t_i choose a point $x_i \in \gamma(t_i)^* \subset \mathcal{X}$; clearly

$$|\iota(x_i) \iota_n \circ \gamma(t_i)|_{\mathbb{E}^d} < 2 \cdot a_n. \quad (***)$$

Note that $x_{i-1}, x_i \in \gamma([t_{i-1}, t_i])^*$. Thus, (**) and (**) imply that

$$|x_{i-1} x_i| < \text{diam } \gamma_n([t_{i-1}, t_i])^* < \varepsilon$$

for all large n . Thus x_i forms an ε -chain from x to x' , and (***) implies

$$\begin{aligned} \text{pull}_{i,\varepsilon}(x, x') &\leq \sum_{i=1}^m |\iota(x_{i-1}) \iota(x_i)| < \\ &< \sum_{i=1}^m |\iota_n \circ \gamma_n(t_{i-1}) \iota_n \circ \gamma_n(t_i)| + 4 \cdot a_n \cdot \lceil \frac{\ell}{\delta} \rceil < \\ &< \ell + 4 \cdot a_n \cdot \lceil \frac{\ell}{\delta} \rceil \end{aligned}$$

which contradicts $(**)$ for large enough n . \square

Remark. In the constructed inverse system $(\varphi_{m,n}, \mathcal{P}_n)$, the images of $\varphi_{m,n}$ form a $\sqrt{d} \cdot a_n$ -net in \mathcal{P}_n . It follows that the space \mathcal{X} is isometric to the Gromov–Hausdorff limit of \mathcal{P}_n (see also section 2).

Proof of proposition 1.7. The inequality $d \leq m$ follow from trivial statement 1.1. In the proof of the second part, we use the following two statements:

1. Assume $\iota: \Omega \subset \mathbb{M}^d \rightarrow \mathbb{E}^m$ is an intrinsic isometry, then it is a Lipschitz map for a Euclidean structure on Ω . Thus, according to Rademacher’s theorem (see [Federer, 3.1.6]) the differential $d_p \iota$ is well defined almost all $p \in \Omega$.
2. For any curve $\gamma(t)$ with natural parameter in a metric space, we have that for almost all values of parameter t_0 we have

$$|\gamma(t_0) \gamma(t_0 + \varepsilon)| = \varepsilon + o(\varepsilon),$$

see [BBI, 2.7.5].

Let us denote by $\|\cdot\|$ the norm which induces metric on \mathbb{M}^d . Fix u so that $\|u\| = 1$. Consider pencil of lines of the form $p + u \cdot t$ in Ω . Two statements above imply that $|d_p \iota(v)| \stackrel{a.e.}{=} \|v\|$. Hence we obtain parallelogram identity

$$2 \cdot (\|u\|^2 + \|v\|^2) = \|u + v\|^2 + \|u - v\|^2$$

is satisfied for any two vectors v and w . I.e. the norm $\|\cdot\|$ is Euclidean. \square

4 About path isometries

In this section we will relate the notion of intrinsic isometry defined in section 2 with more common (but less natural) notion of path and weak path isometries.

4.1. Definition. *Let \mathcal{X} and \mathcal{Y} be two length spaces. A map $\iota: \mathcal{X} \rightarrow \mathcal{Y}$ is called*

1. *path isometry if for any path $\gamma: [0, 1] \rightarrow \mathcal{X}$ we have*

$$\text{length } \gamma = \text{length } \iota \circ \gamma.$$

2. *weak path isometry if for any rectifiable path $\gamma: [0, 1] \rightarrow \mathcal{X}$ we have*

$$\text{length } \gamma = \text{length } \iota \circ \gamma.$$

As it was noted in section 2, any intrinsic isometry is a path isometry (and therefore, a weak path isometry). Next we will show that converse does not hold. Similar counterexamples for weak path isometries are much simpler: one can take a left-invariant sub-Riemannian metric d on Heisenberg group H then factorizing by center gives an weak path isometry $(H, d) \rightarrow \mathbb{E}^2$ (which is not a path isometry and thus not an intrinsic isometry).

4.2. Example. *There is a length space \mathcal{X} and a path isometry $f: \mathcal{X} \rightarrow \mathbb{R}$ such that $f^{-1}(0)$ is connected nontrivial subset.*

Moreover, in such example the Lebesgue covering dimension of $f^{-1}(0)$ can be made arbitrary large.

In particular, an analog of 1.1 does not hold for path isometries.

The following construction was suggested by D. Burago; it is based on two ideas: (1) the construction in [BIS, 3.1], (2) the construction of *pseudo-arc* given in [Knaster] (see also the survey [Lewis] and references therein). In fact, for the first part of theorem $f^{-1}(0)$ will be homeomorphic to a pseudo-arc and for the second part $f^{-1}(0)$ will be homeomorphic to a product of pseudo-arcs.

Proof. The space \mathcal{X} will be a completion $\bar{\Gamma}$ of certain metric graph⁵ Γ .

First let us describe the construction of f modulo a construction of Γ . Set $\dot{\Gamma} = \bar{\Gamma} \setminus \Gamma$. Consider map $f: \bar{\Gamma} \rightarrow \mathbb{R}$, where $f(x)$ is the distance from x to $\dot{\Gamma}$. Then f is a path isometry on Γ and $f(\dot{\Gamma}) = 0$. To finish that proof we will have to construct Γ on such a way that

- (i) $\dot{\Gamma}$ is connected and contains more than one point;
- (ii) f is a path isometry on whole $\bar{\Gamma}$ (not only on Γ).

Construction of Γ . For two real intervals \mathbb{I} and \mathbb{J} , a continuous onto map $h: \mathbb{I} \rightarrow \mathbb{J}$ will be called ε -crooked if for any two values $t_1 < t_2$ in \mathbb{I} there are values $t_1 < t'_2 < t'_1 < t_2$ such that $|h(t'_i) - h(t_i)| \leq \varepsilon$ for $i \in \{1, 2\}$. The existence of ε -crooked map for any given \mathbb{I} and $\varepsilon > 0$ is easy to prove by induction on $n = \lceil \frac{1}{\varepsilon} \cdot \text{length } \mathbb{J} \rceil$.

Let us fix a sequence of real intervals \mathbb{J}_n with short $\frac{1}{2^n}$ -crooked maps $h_n: \mathbb{J}_n \rightarrow \mathbb{J}_{n-1}$. The topological inverse limit $\mathbb{J}_\infty = \varprojlim \mathbb{J}_n$ is a connected compact which has no nontrivial paths.

We can think of \mathbb{J}_n as a (linear) metric graph with length of each edge $\leq \frac{1}{2^n}$. Construct a graph Γ from a disjoint union $\sqcup_n \mathbb{J}_n$ by joining each vertex v of \mathbb{J}_n to a vertex of \mathbb{J}_{n-1} which is closest to $h_n(v)$ by an edge of length $\frac{1}{2^n}$. Then $\dot{\Gamma}$ is homeomorphic to \mathbb{J}_∞ ; thus we get (i).

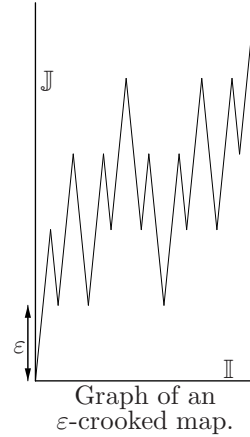
Let us denote by Γ_n the finite subgraph of Γ formed by all vertexes in $\mathbb{J}_1, \mathbb{J}_2, \dots, \mathbb{J}_n$. Note that there is a short map $\Gamma_n \rightarrow \Gamma_{n-1}$ which is identity on Γ_{n-1} . It follows that for any path $\alpha: [0, 1] \rightarrow \bar{\Gamma}$ we have that the total length of $\alpha \setminus \dot{\Gamma}$ is at least $|\alpha(0) - \alpha(1)|_{\bar{\Gamma}}$. Thus (ii) follows.

Second part. We construct a graph $\Gamma^{(m)}$ to make $\dot{\Gamma}^{(m)}$ homeomorphic to a product on m copies of $\dot{\Gamma}$.

We will do the case $m = 2$; the others are analogous. The set of vertexes of $\Gamma^{(2)}$ is disjoint union $\sqcup_n (\text{Vert } \mathbb{J}_n \times \text{Vert } \mathbb{J}_n)$, where $\text{Vert } \mathbb{J}_n$ denotes the set of vertexes of \mathbb{J}_n . We connect two vertexes $(x, y) \in \text{Vert } \mathbb{J}_n \times \text{Vert } \mathbb{J}_n$ and $(x', y') \in \text{Vert } \mathbb{J}_k \times \text{Vert } \mathbb{J}_k$ iff the pairs (x, x') and (y, y') were connected in Γ ; the length of this edge must be maximum of lengths of edges xx' and yy' (we assume that a vertex is connected to it-self by an edge of length 0).

Clearly, there is a homeomorphism $\dot{\Gamma}^{(2)} \rightarrow \dot{\Gamma} \times \dot{\Gamma}$. Note that there are two short coordinate projections $\varsigma_1, \varsigma_2: \Gamma^{(2)} \rightarrow \Gamma$. Thus for any path $\alpha: [0, 1] \rightarrow \bar{\Gamma}^2$,

⁵i.e., locally finite graph with intrinsic metric, such that each edge is isometric to a real interval.



we have that total length of $\alpha \setminus \dot{\Gamma}$ is at least as big as $\max_i |\zeta_i \circ \alpha(0) \zeta_i \circ \alpha(1)|$. That ensures that pulled back metric on $\dot{\Gamma}^{(2)}$ is bi-Lipschitz to the product metric on $\dot{\Gamma} \times \dot{\Gamma}$. \square

5 Comments and open questions

A length space \mathcal{M} is called *Minkowski d -polyhedron* if there is a finite triangulation of \mathcal{M} such that each simplex is isometric to a simplex in a Minkowski space. Correspondingly, a compact metric space \mathcal{X} is called *pro-Minkowski space* of rank $\leq d$ if it can be presented as an inverse limit of Minkowski d -polyhedra.

5.1. Question. *Is it true that any length space with Lebesgue's covering dimension d is a pro-Minkowski space of rank d ?*

Or even more specific:

5.2. Question. *Is it true that any metric space which homeomorphic to a disk is a pro-Minkowski space of rank 2?*

One can reformulate it philosophically: *Is there any essential difference between Finsler metric and general metric on n -manifold?* This question was asked by D. Burago; it was also original motivation for this paper (see also a related example [BIS, theorem 1]).

If one removes restriction on dimension, then the answer to the above question is YES. Namely, the following exercise can be solved by using Kuratowski embedding $x \mapsto \text{dist}_x$.

5.3. Exercise. *Show that any compact length space is an inverse limit of Minkowski polyhedra \mathcal{M}_n with $\dim \mathcal{M}_n \rightarrow \infty$.*

5.4. Question. *Is it true that any path isometry from a closed Euclidean ball to Euclidean space is an intrinsic isometry?*

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