

# On intrinsic isometries to Euclidean space.

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## Abstract

I consider length spaces which admit intrinsic isometries to Euclidean  $d$ -space. The main result roughly states that the class of these spaces coincides with class of inverse limits of Euclidean  $d$ -polyhedra.

The proof is straight forward, but I like the statement; it is funny that one can use inverse limits to describe a natural class of metric spaces.

## 1 Introduction

The definition of intrinsic isometry is given in section 2. That is a bit stronger version of path isometry (see 4.1) (in particular, any intrinsic isometry preserves lengths of curves). The following statement is one of the reason we prefer intrinsic isometry.

**1.1. Trivial statement.** *If a compact length space  $\mathcal{X}$  admits an intrinsic isometry to  $d$ -dimensional Euclidean space (further denoted by  $\mathbb{E}^d$ ) then*

$$\dim \mathcal{X} \leq d,$$

where  $\dim$  denotes the Lebesgue's covering dimension.

An analogous statement for path isometry does not hold, see example 4.2.

The Hausdorff dimension can not be bounded this way. For example,  $\mathbb{R}$ -tree admits an intrinsic isometry to  $\mathbb{R}$  and it contains compact subsets of arbitrary large Hausdorff dimension.

Here are some known results on length spaces which admit *intrinsic isometry* to  $\mathbb{E}^d$ .

**1.2. Theorem.** *Let  $\mathcal{R}$  be  $d$ -dimensional Riemannian space and  $f: \mathcal{R} \rightarrow \mathbb{E}^d$  be a short map<sup>1</sup>. Then given  $\varepsilon > 0$ , there is an intrinsic isometry  $\iota: \mathcal{R} \rightarrow \mathbb{E}^d$  such that*

$$|f(x) \iota(x)|_{\mathbb{E}^d} < \varepsilon$$

for any  $x \in \mathcal{R}$ .

*In particular, any Riemannian  $d$ -space admits an intrinsic isometry to  $\mathbb{E}^d$ .*

For path isometries, this theorem was proved in [Gromov, 2.4.11], and the same proof works for intrinsic isometries. Applying this theorem, one can show that any limit of increasing sequence of Riemannian metrics on a fixed  $d$ -dimensional manifold admits an intrinsic isometry to  $\mathbb{E}^d$ . (The proof is similar to “if”-part of theorem 1.5.) In particular, any sub-Riemannian metric on  $d$ -dimensional manifold admits an intrinsic isometry to  $\mathbb{E}^d$ .

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<sup>1</sup>i.e. 1-Lipschitz map

**1.3. Theorem.** *Let  $\mathcal{P}$  be a Euclidean polyhedron and  $f: \mathcal{P} \rightarrow \mathbb{E}^d$  be a short map. Then, given  $\varepsilon > 0$ , there is a piecewise linear intrinsic isometry  $\iota: \mathcal{P} \rightarrow \mathbb{E}^d$  such that*

$$|f(x)\iota(x)|_{\mathbb{E}^d} < \varepsilon$$

for any  $x \in \mathcal{P}$ .

**1.4. Corollary.** *Any  $d$ -dimensional Euclidean polyhedron admits a piecewise linear intrinsic isometry to  $\mathbb{E}^d$ .*

The corollary was proved in [Zalgaller] for dimension  $\leq 4$ , but a slight modification of the proof works in all dimensions, see [Krat]. The 2-dimensional case of this theorem was proved in [Krat]. Later, the proof was extended to all dimensions in [Akopyan] using surprisingly quite a trick discovered in [Brehm] (The paper of Brehm was left without attention for many years. The main idea in this paper was first rediscovered in [Akopyan–Tarasov] and only recently the above reference was found).

**Iff-condition.** Now we describe the main result of the paper.

A compact length space  $\mathcal{X}$  is called *pro-Euclidean space of rank  $\leq d$*  if it can be presented as an *inverse limit*  $\mathcal{X} = \varprojlim \mathcal{P}_n$  (see section 2) of a sequence of Euclidean  $d$ -polyhedra  $\mathcal{P}_n$ .

**1.5. Theorem.** *A compact length space  $\mathcal{X}$  admits an intrinsic isometry to  $\mathbb{E}^d$  if and only if  $\mathcal{X}$  is a pro-Euclidean space of rank  $\leq d$ .*

Thus, statement in theorem 1.2 is equivalent to the fact that any Riemannian  $d$ -manifold is a pro-Euclidean space of rank  $\leq d$ . The latter can be obtained directly from the following exercise; this way theorem 1.5 provides an alternative proof to theorem 1.2.

**1.6. Exercise.** *Show that any compact Riemannian manifold admits a Lipschitz approximation by Euclidean polyhedra.*

**A non-example.** Let us remind that *Minkowski space* has nothing to do with space-time; it is finite dimensional real vector spaces with metric induced by a norm.

**1.7. Theorem.** *Let  $\Omega$  be an open subset of Minkowski  $d$ -space  $\mathbb{M}^d$ . Assume  $\Omega$  admits an intrinsic<sup>2</sup> isometry to  $\mathbb{E}^d$  then  $\mathbb{M}^d$  is isometric to a  $\mathbb{E}^d$ .*

In particular, the condition 1.1 on Lebesgue’s dimension is not sufficient.

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## 2 Preliminaries

**Standard definitions.** Given a metric space  $\mathcal{X}$  and two points  $x, x' \in \mathcal{X}$ , we will denote by  $|xx'| = |xx'|_{\mathcal{X}}$  the distance from  $x$  to  $x'$  in  $\mathcal{X}$ .

A *length space* is a metric space such that for any two points  $x, x'$  the distance  $|xx'|$  coincides with the infimum of lengths of curves connecting  $x$  and  $x'$ .

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<sup>2</sup>In fact the same is true for path isometry.

A map  $f: \mathcal{X} \rightarrow \mathcal{Y}$  between metric spaces  $\mathcal{X}$  and  $\mathcal{Y}$  is called *short* if it is 1-Lipschitz, i.e. if for any  $x, x' \in \mathcal{X}$  we have

$$|f(x)f(x')|_{\mathcal{Y}} \leq |xx'|_{\mathcal{X}}.$$

A length space  $\mathcal{P}$  is called *Euclidean  $d$ -polyhedron* if there is a finite triangulation of  $\mathcal{P}$  such that each simplex is isometric to a simplex in  $\mathbb{E}^d$ .

**Inverse limit.** Consider an *inverse system* of compact length spaces  $(\mathcal{X}_n)_{n=0}^{\infty}$  and short maps  $\varphi_{m,n}: \mathcal{X}_m \rightarrow \mathcal{X}_n$  for  $m \geq n$ ; i.e.,

1.  $\varphi_{m,n} \circ \varphi_{k,m} = \varphi_{k,n}$  for any triple  $k \geq m \geq n$  and
2. for any  $n$ , the map  $\varphi_{n,n}$  is identity map of  $\mathcal{X}_n$ .

A length space  $\mathcal{X}$  is called *inverse limit of the system*  $(\varphi_{m,n}, \mathcal{X}_n)$  (denoted by  $\mathcal{X} = \varprojlim \mathcal{X}_n$ ) if its underlying space consists of all sequences  $x_n \in \mathcal{X}_n$  such that  $\varphi_{m,n}(x_m) = x_n$  for all  $m \geq n$  and for two such sequences  $(x_n)$  and  $(x'_n)$  the distance is defined by

$$|(x_n)(x'_n)|_{\mathcal{X}} = \lim_{n \rightarrow \infty} |x_n x'_n|_{\mathcal{X}_n}.$$

If  $\mathcal{X} = \varprojlim \mathcal{X}_n$ , then the map  $\psi_n: \mathcal{X} \rightarrow \mathcal{X}_n$ , defined by  $\psi_n: (x_i)_{i=0}^{\infty} \mapsto x_n$  are called *projections*. Clearly  $\psi_n = \varphi_{m,n} \circ \psi_m$  for all  $m \geq n$ .

*Comments.* The above definition is equivalent to the usual inverse limit in the category with class of objects formed by compact length spaces and class of morphisms by short maps.

**Intrinsic isometries and pull back metrics.** Let  $f: \mathcal{X} \rightarrow \mathcal{Y}$  be a short map between length spaces  $\mathcal{X}$  and  $\mathcal{Y}$ . Given two points  $x, x' \in \mathcal{X}$ , a sequence of points  $x = x_0, x_1, \dots, x_n = x'$  is called  $\varepsilon$ -chain from  $x$  to  $x'$  if  $|x_{i-1}x_i| \leq \varepsilon$  for all  $i > 0$ . Set

$$\text{pull}_{f,\varepsilon}(x, x') = \inf \left\{ \sum_{i=1}^n |f(x_{i-1})f(x_i)|_{\mathcal{Y}} \right\}$$

where the infimum is taken along all  $\varepsilon$ -chains  $(x_i)_{i=0}^n$  from  $x$  to  $x'$ .

Clearly  $\text{pull}_{f,\varepsilon}$  is a pre-metric<sup>3</sup> on  $\mathcal{X}$ ,  $\text{pull}_{f,\varepsilon}(x, x') \leq |xx'|_{\mathcal{X}}$  and  $\text{pull}_{f,\varepsilon}(x, x')$  is non-increasing in  $\varepsilon$ . Thus, the following limit

$$\text{pull}_f(x, x') = \lim_{\varepsilon \rightarrow 0} \text{pull}_{f,\varepsilon}(x, x')$$

is well defined. The pre-metric  $\text{pull}_f$  on  $\mathcal{X}$  will be called *pull back metric* for  $f$ .

We say that a short map  $f: \mathcal{X} \rightarrow \mathcal{Y}$  between length spaces  $\mathcal{X}$  and  $\mathcal{Y}$  is an *intrinsic isometry* if

$$|xx'|_{\mathcal{X}} = \text{pull}_f(x, x')$$

for any  $x, x' \in \mathcal{X}$ .

It is easy to see that intrinsic isometry preserves the lengths of curves. The converse does not hold, see section 4.

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<sup>3</sup>i.e. it satisfies triangle inequality, it is symmetric, non-negative and  $\text{pull}_{f,\varepsilon}(x, x) = 0$ , but it might happen that  $\text{pull}_{f,\varepsilon}(x, x') = 0$  for  $x \neq x'$ .

**2.1. Proposition.** *Let  $\mathcal{X}$  and  $\mathcal{Y}$  be length spaces,  $\mathcal{X}$  be compact and  $f: \mathcal{X} \rightarrow \mathcal{Y}$  be a short map. Then given  $\varepsilon > 0$  there is  $\delta = \delta(f, \varepsilon) > 0$  such that for any short map  $h: \mathcal{X} \rightarrow \mathcal{Y}$  such that*

$$|f(x)h(x)|_{\mathcal{Y}} < \delta \text{ for any } x \in \mathcal{X}$$

*we have*

$$\text{pull}_f(x, x') < \text{pull}_h(x, x') + \varepsilon$$

*for any  $x, x' \in \mathcal{X}$ .*

The proof is a direct application of the lemma 2.2.

For a compact metric space  $\mathcal{X}$ , we denote by  $\text{pack}_\varepsilon \mathcal{X}$  the maximal number of points in  $\mathcal{X}$  on distance  $> \varepsilon$ . Clearly  $\text{pack}_\varepsilon \mathcal{X}$  is finite for any  $\varepsilon > 0$ .

**2.2. Lemma.** *Let  $\mathcal{X}$  and  $\mathcal{Y}$  be a length spaces,  $\mathcal{X}$  is compact and  $f, h: \mathcal{X} \rightarrow \mathcal{Y}$  be two short maps.*

*Assume for any  $x \in \mathcal{X}$ ,  $|f(x)h(x)| < \delta$  then for any  $x, x' \in \mathcal{X}$  we have*

$$\text{pull}_{f,\varepsilon}(x, x') \leq \text{pull}_{h,\varepsilon}(x, x') + 4 \cdot \delta \cdot \text{pack}_\varepsilon \mathcal{X}.$$

*Proof.* Assume  $\text{pull}_{h,\varepsilon}(x, x') < \ell$ , i.e. there is an  $\varepsilon$ -chain  $\{x_i\}_{i=0}^n$  from  $x$  to  $x'$  such that

$$\sum_{i=1}^n |h(x_{i-1})h(x_i)|_{\mathcal{Y}} < \ell. \quad (*)$$

Since  $|h(x_i)f(x_i)| < \delta$ ,

$$\text{pull}_{f,\varepsilon}(x, x') \leq \sum_{i=1}^n |f(x_{i-1})f(x_i)|_{\mathcal{Y}} < \sum_{i=1}^n |h(x_{i-1})h(x_i)|_{\mathcal{Y}} + 2n\delta$$

Assume  $n$  is the smallest number for which there is an  $\varepsilon$ -chain satisfying (\*). It is enough to show that

$$n < 2 \cdot \text{pack}_\varepsilon \mathcal{X}.$$

If  $n \geq 2 \text{pack}_\varepsilon \mathcal{X}$ , there are  $i$  and  $j$  such that  $j-i > 1$  and  $|x_i x_j| \leq \varepsilon$ . Remove from this chain all elements  $x_k$  with  $i < k < j$ ; i.e. consider new  $\varepsilon$ -chain

$$x = x_0, \dots, x_{i-1}, x_i, x_j, x_{j+1}, \dots, x_n = x'$$

By triangle inequality in  $\mathcal{Y}$ , the new chain satisfies (\*); i.e.  $n$  is not the smallest number, a contradiction.  $\square$

**2.3. Proposition.** *Let  $\mathcal{X}$  and  $\mathcal{Y}$  be length spaces,  $\mathcal{X}$  be compact and  $\iota: \mathcal{X} \rightarrow \mathcal{Y}$  be an intrinsic isometry.*

*Then given  $\varepsilon > 0$  there is  $\delta = \delta(\iota, \varepsilon) > 0$  such that for any connected set  $W \subset \mathcal{X}$*

$$\text{diam } \iota(W) < \delta \implies \text{diam } W < \varepsilon.$$

*Proof.* Assume contrary, i.e. there is a sequence of connected subsets  $W_n \subset \mathcal{X}$  such that  $\text{diam } \iota(W_n) \rightarrow 0$  as  $n \rightarrow \infty$  but  $\text{diam } W_n > \varepsilon$ . Thus there are two sequences of points  $x_n, x'_n \in W_n$  such that  $|x_n x'_n| \geq \varepsilon$ . Pass to a subsequence

of  $n$  so that  $W_n \rightarrow W$  in Hausdorff sense and  $x_n \rightarrow x$ ,  $x'_n \rightarrow x'$ . We obtain a closed connected subset  $W \subset \mathcal{X}$  with two distinct points  $x$  and  $x'$  such that  $\iota(W) = p$  for some  $p \in \mathcal{Y}$ .

Since  $W$  is connected, for any  $\varepsilon > 0$  there is an  $\varepsilon$ -chain  $(x_i)_{i=0}^n$  from  $x$  to  $x'$  such that  $\iota(x_i) = p$  for all  $i$ . Thus, we have  $\text{pull}_{\iota, \varepsilon}(x, x') = 0$  for any  $\varepsilon > 0$ ; i.e.,  $\text{pull}_\iota(x, x') = 0$ , a contradiction.  $\square$

### 3 The proofs

*Proof of the trivial statement (1.1).* Given  $\varepsilon > 0$  choose  $\delta = \delta(\iota, \varepsilon)$  as in proposition 2.3. Since  $\dim \mathbb{E}^d = d$ , for any  $\delta > 0$  there is a finite open covering  $\{U_i\}_{i=1}^n$  of  $\iota(\mathcal{X})$  with multiplicity  $\leq d + 1$  and such that  $\text{diam } U_i < \delta$  for each  $i$ .

Consider the covering  $\{V_\alpha\}$  of  $\mathcal{X}$  by connected components of  $\iota^{-1}(U_i)$  for all  $i$ . According to proposition 2.3,  $\text{diam } V_\alpha < \varepsilon$ . Clearly multiplicity of  $\{V_\alpha\}$  is at most  $d + 1$ . Thus, the statement follows.  $\square$

*Proof of “if” in 1.5.* Let  $\mathcal{X}$  be a pro-Euclidean space of rank  $\leq d$ . Assume  $(\mathcal{P}_n)_{n=0}^\infty$  is a sequence of  $d$ -dimensional Euclidean polyhedra and

$$\varphi_{m,n}: \mathcal{P}_m \rightarrow \mathcal{P}_n$$

is an inverse system of short maps such that  $\mathcal{X} = \varprojlim \mathcal{P}_n$ . Let  $\psi_n: \mathcal{X} \rightarrow \mathcal{P}_n$  be the projections.

According to theorem 1.3, given  $\varepsilon_{n+1} > 0$  and a picewise linear intrinsic isometry  $\iota_n: \mathcal{P}_n \rightarrow \mathbb{E}^d$  there is a picewise linear intrinsic isometry  $\iota_{n+1}: \mathcal{P}_{n+1} \rightarrow \mathbb{E}^d$  such that for any  $x \in \mathcal{P}_n$  and

$$|\iota_{n+1}(x) \iota_n \circ \varphi_{n+1,n}(x)| < \varepsilon_{n+1}.$$

It remains to show that sequence  $\varepsilon_n$  can be chosen on such a way that  $\iota_n \circ \psi_n$  converges to an isometry  $\iota: \mathcal{X} \rightarrow \mathbb{E}^d$ .

Let us choose  $\varepsilon_{n+1} > 0$  so that

$$\varepsilon_{n+1} < \frac{1}{2} \min\{\varepsilon_n, \delta(\iota_n, \frac{1}{n})\},$$

where  $\delta(\iota_n, \frac{1}{n})$  as in proposition 2.1.

Choosing sequence  $\varepsilon_n$  this way, it is clear that  $\sum_i \varepsilon_i < \infty$ , thus the the following limit exists

$$\iota = \lim_{n \rightarrow \infty} \iota_n \circ \psi_n, \quad \iota: \mathcal{X} \rightarrow \mathbb{E}^d.$$

Clearly  $\iota$  is short. Further, for any  $x \in \mathcal{X}$ ,

$$|\iota(x) \iota_n \circ \psi_n(x)| < \sum_{i=n+1}^{\infty} \varepsilon_i < \delta(\iota_n, \frac{1}{n}).$$

Thus, according to proposition 2.1,

$$\text{pull}_\iota(x, x') + \frac{1}{n} > \text{pull}_{\iota_n \circ \psi_n}(x, x') \geq |\psi_n(x) \psi_n(x')|_{\mathcal{P}_n}.$$

Since  $|\psi_n(x) \psi_n(x')|_{\mathcal{P}_n} \rightarrow |xx'|_{\mathcal{X}}$  as  $n \rightarrow \infty$ , the map  $\iota: \mathcal{X} \rightarrow \mathbb{E}^d$  is an intrinsic isometry.  $\square$

*Proof of “only if” in 1.5.* We will give a construction a polyhedron  $\mathcal{P}$  associated to an intrinsic isometry  $\iota: \mathcal{X} \rightarrow \mathbb{E}^d$  and a tiling of  $\mathbb{E}^d$  by coordinate  $a$ -cubes. (The space  $\mathcal{P}$  will be glued out of  $a$ -cubes.) The construction will be done in such a way that if a tiling  $\tau'$  is a subdivision of a tiling  $\tau$  then for corresponding polyhedra  $\mathcal{P}'$  and  $\mathcal{P}$  there will be a natural intrinsic isometry  $\mathcal{P}' \rightarrow \mathcal{P}$ . Thus we will construct the needed inverse system of polyhedra out of nested subdivisions of  $\mathbb{E}^d$ .

Take sequences  $a_n = \frac{1}{2^n}$  and set  $r_n = \frac{1}{10} \cdot a_n$ . Fix  $n$  for a while and consider tiling of  $\mathbb{E}^d$  by coordinate  $a_n$ -cubes. Let us construct a Euclidean polyhedron  $\mathcal{P}_n$  associated to this tiling.

The image  $\iota(\mathcal{X})$  is covered by finite number of such  $a_n$ -cubes  $\{\square_n^i\}$ . For each  $\square_n^i$ , consider all connected components  $\{W_n^{ij}\}$  of

$$B_{r_n}(\iota^{-1}(\square_n^i)) \subset \mathcal{X},$$

where  $B_r(S)$  denotes  $r$ -neighborhood of set  $S$ .

For fixed  $i$ , the collection of open sets  $\{W_n^{ij}\}$  is finite since  $W_n^{ij}$  are disjoint and each contains an  $r_n$ -ball in  $\mathcal{X}$ . Thus, the set of all  $\{W_n^{ij}\}$  for all  $\{\square_n^i\}$  forms a finite open cover of  $\mathcal{X}$ . For each  $W_n^{ij}$  make an isometric copy  $\square_n^{ij}$  of  $\square_n^i$  and fix an isometry  $\iota_n^{ij}: \square_n^{ij} \rightarrow \square_n^i$ . The Euclidean polyhedron  $\mathcal{P}_n$ , is glued from  $\square_n^{ij}$  by the following rule: glue  $\square_n^{i_1 j_1}$  to  $\square_n^{i_2 j_2}$  along  $(\iota_n^{i_2 j_2})^{-1} \circ \iota_n^{i_1 j_1}$  iff  $W_n^{i_1 j_1} \cap W_n^{i_2 j_2} \neq \emptyset$ . (The map  $(\iota_n^{i_2 j_2})^{-1} \circ \iota_n^{i_1 j_1}$  sends one of the faces of  $\square_n^{i_1 j_1}$  isometrically to a face of  $\square_n^{i_2 j_2}$ .)

The constructed polyhedron  $\mathcal{P}_n$  admits a natural piecewise linear intrinsic isometry  $\iota_n: \mathcal{P}_n \rightarrow \mathbb{E}^d$ , defined as  $\iota_n(x) = \iota_n^{ij}(x)$  if  $x \in \square_n^{ij}$ . Further, there is uniquely defined intrinsic isometry  $\varphi_{m,n}: \mathcal{P}_m \rightarrow \mathcal{P}_n$  for  $m \geq n$  which satisfies  $\iota_m = \iota_n \circ \varphi_{m,n}$  and

$$\varphi_{m,n}(\square_m^{i'j'}) \subset \square_n^{ij} \subset \mathcal{P}_n \Rightarrow W_m^{i'j'} \subset W_n^{ij} \subset \mathcal{X}.$$

One can define intrinsic isometries  $\psi_n: \mathcal{X} \rightarrow \mathcal{P}_n$  on a way similar to  $\varphi_{n,m}$ . Namely,  $\psi_n: \mathcal{X} \rightarrow \mathcal{P}_n$  is uniquely determined by  $\iota_n \circ \psi_n = \iota$  and

$$\psi_n(x) \in \square_n^{ij} \subset \mathcal{P}_n \Rightarrow x \in W_n^{ij} \subset \mathcal{X}.$$

Clearly,  $\mathcal{P}_n$  together with  $\varphi_{m,n}$  form an inverse system and  $\psi_n = \varphi_{m,n} \circ \psi_m$  for any pair  $m \geq n$ .

In order to prove that  $\mathcal{X} = \varprojlim \mathcal{P}_n$ , it only remains to show that

$$|xx'|_{\mathcal{X}} \leq \lim_{n \rightarrow \infty} |\psi_n(x) \psi_n(x')| \quad (*)$$

for all  $x, x' \in \mathcal{X}$ .

Given a subset  $K \subset \mathcal{P}_n$ , let us denote by  $K^* \subset \mathcal{X}$  the union of all  $W_n^{ij} \subset \mathcal{X}$  such that  $\square_n^{ij} \cap K \neq \emptyset$ . Clearly, if  $K$  is connected then so is  $K^*$ . More over,  $\iota(K^*) \subset B_{r_n}(\iota_n(K))$ . Thus, from proposition 2.3, we have that for any  $\varepsilon > 0$  we can find  $\delta > 0$  such that

$$r_n + \text{diam } K < \delta \implies \text{diam } K^* < \varepsilon \quad (**)$$

Assume (\*) is wrong, then one can choose  $x, x' \in \mathcal{X}$  and  $\varepsilon, \ell > 0$  so that

$$\text{pull}_{i,\varepsilon}(x, x') > \ell > |\psi_n(x) \psi_n(x')|_{\mathcal{P}_n} \quad (**)$$

for all  $n$ . In particular, for all  $n$  there is a path  $\gamma_n: [0, 1] \rightarrow \mathcal{P}_n$  from  $\psi_n(x)$  to  $\psi_n(x')$  with length  $< \ell$ . Choose  $\delta = \delta(\varepsilon, \ell)$  as in proposition 2.3. Let  $0 = t_0 < t_1 < \dots < t_m = 1$  be such that

$$\text{diam } \gamma([t_{i-1}, t_i]) < \frac{\delta}{2}. \quad (**)$$

Clearly one can assume that  $m \leq 2 \cdot \lceil \frac{\ell}{\delta} \rceil$ . For each  $t_i$  choose a point  $x_i \in \gamma(t_i)^* \subset \mathcal{X}$ ; clearly

$$|\iota(x_i) \iota_n \circ \gamma(t_i)|_{\mathbb{E}^d} < 2 \cdot a_n. \quad (***)$$

Note that  $x_{i-1}, x_i \in \gamma([t_{i-1}, t_i])^*$ . Thus, (\*\*) and (\*\*\*) imply that

$$|x_{i-1} x_i| < \text{diam } \gamma_n([t_{i-1}, t_i])^* < \varepsilon$$

for all large  $n$ . Thus  $x_i$  forms an  $\varepsilon$ -chain from  $x$  to  $x'$ , and (\*\*\*) implies

$$\begin{aligned} \text{pull}_{i,\varepsilon}(x, x') &\leq \sum_{i=1}^m |\iota(x_{i-1}) \iota(x_i)| < \\ &< \sum_{i=1}^m |\iota_n \circ \gamma_n(t_{i-1}) \iota_n \circ \gamma_n(t_i)| + 4 \cdot a_n \cdot \lceil \frac{\ell}{\delta} \rceil < \\ &< \ell + 4 \cdot a_n \cdot \lceil \frac{\ell}{\delta} \rceil \end{aligned}$$

which contradicts (\*\*) for large enough  $n$ .  $\square$

*Proof of 1.7.* Let us denote by  $\|\cdot\|$  the norm which induces metric on  $\mathbb{M}^d$ .

Assume  $\iota: \Omega \subset \mathbb{M}^d \rightarrow \mathbb{E}^d$  is an intrinsic isometry, then it is a Lipschitz map for a Euclidean structure on  $\Omega$ . Thus, according to Rademacher's theorem (see [Federer, 3.1.6]) the differential  $d_p \iota$  is well defined almost all  $p \in \Omega$ .

Further, for any curve  $\gamma(t)$  with natural parameter in a metric space, we have that for almost all values of parameter  $t_0$  we have

$$|\gamma(t_0) \gamma(t_0 + \varepsilon)| = \varepsilon + o(\varepsilon),$$

see [BBI, 2.7.5].

Given two vectors  $u, v$  in  $\mathbb{M}^d$  one can apply both statements to the net formed by four families of lines; each is parallel to  $u, v, u+v$  and  $u-v$  correspondingly. We get that norms  $\|u\|, \|v\|, \|u+v\|$  and  $\|u-v\|$  satisfy parallelogram rule; i.e.

$$2 \cdot (\|u\|^2 + \|v\|^2) = \|u+v\|^2 + \|u-v\|^2.$$

Thus,  $\mathbb{M}^d$  is isometric to Euclidean space.  $\square$

## 4 About path isometries

In this section we will relate the notion of intrinsic isometry defined in section 2 with more common (but less natural) notion of path and weak path isometries.

**4.1. Definition.** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be two length spaces. A map  $\iota: \mathcal{X} \rightarrow \mathcal{Y}$  is called

1. path isometry if for any path  $\gamma: [0, 1] \rightarrow \mathcal{X}$  we have

$$\text{length } \gamma = \text{length } \iota \circ \gamma.$$

2. weak path isometry if for any rectifiable path  $\gamma: [0, 1] \rightarrow \mathcal{X}$  we have

$$\text{length } \gamma = \text{length } \iota \circ \gamma.$$

As it was noted in section 2, any intrinsic isometry is a path isometry (and therefore, a weak path isometry). Next we will show that reverse does not hold. Similar counterexamples for weak path isometries are much simpler: one can take a left-invariant sub-Riemannian metric  $d$  on Heisenberg group  $H$  then factorizing by center gives an weak path isometry  $(H, d) \rightarrow \mathbb{E}^2$  (which is not a path isometry and thus not an intrinsic isometry).

**4.2. Example.** *There is a length space  $\mathcal{X}$  and a path isometry  $f: \mathcal{X} \rightarrow \mathbb{R}$  such that  $f^{-1}(0)$  is connected nontrivial subset.*

*Moreover, in such example the Lebesgue covering dimension of  $f^{-1}(0)$  can be made arbitrary large.*

In particular, an analog of 1.1 does not hold for path isometries.

The following construction was suggested by D. Burago; it is based on two ideas: (1) the construction in [BIS, 3.1], (2) the construction of *Knaster's pseudo-arc* (see [Lewis] and references there in). In fact, for the first part of theorem  $f^{-1}(0)$  will be homeomorphic to a pseudo-arc and for the second part  $f^{-1}(0)$  will be homeomorphic to a product of pseudo-arcs.

*Proof.* The space  $\mathcal{X}$  will be a completion  $\bar{\Gamma}$  of certain length-metric graph  $\Gamma$ .

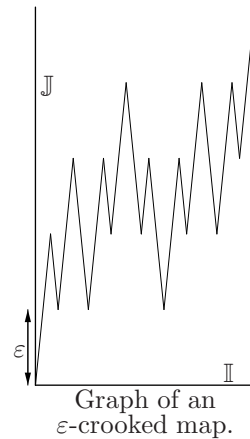
First let us describe the construction of  $f$  modulo a construction of  $\Gamma$ . Set  $\dot{\Gamma} = \bar{\Gamma} \setminus \Gamma$ . Consider map  $f: \bar{\Gamma} \rightarrow \mathbb{R}$ , where  $f(x)$  is the distance from  $x$  to  $\dot{\Gamma}$ . Then  $f$  is a path isometry on  $\Gamma$  and  $f(\dot{\Gamma}) = 0$ . To finish that proof we will have to construct  $\Gamma$  on such a way that

- (i)  $\dot{\Gamma}$  is connected and contains more than one point;
- (ii)  $f$  is a path isometry on whole  $\bar{\Gamma}$  (not only on  $\Gamma$ ).

*Construction of  $\Gamma$ .* For two real intervals  $\mathbb{I}$  and  $\mathbb{J}$ , a continuous onto map  $h: \mathbb{I} \rightarrow \mathbb{J}$  will be called  $\varepsilon$ -crooked if for any two values  $t_1 < t_2$  in  $\mathbb{I}$  there are values  $t_1 < t'_2 < t'_1 < t_2$  such that  $|h(t'_i) - h(t_i)| \leq \varepsilon$  for  $i \in \{1, 2\}$ . The existence of  $\varepsilon$ -crooked map for any given  $\mathbb{I}$  and  $\varepsilon > 0$  is easy to prove by induction on  $n = \lceil \frac{1}{\varepsilon} \cdot \text{length } \mathbb{I} \rceil$ .

Let us fix a sequence of real intervals  $\mathbb{J}_n$  with short  $\frac{1}{2^n}$ -crooked maps  $h_n: \mathbb{J}_n \rightarrow \mathbb{J}_{n-1}$ . The topological inverse limit  $\varprojlim \mathbb{J}_n$  forms a pseudo-arc; in particular it is connected and has no nontrivial paths inside.

We can think of  $\mathbb{J}_n$  as a (linear) metric graph with length of each edge  $\leq \frac{1}{2^n}$ . Construct a graph  $\Gamma$  from a disjointed union  $\sqcup_n \mathbb{J}_n$  by joining each vertex  $v$  of  $\mathbb{J}_n$  to a vertex of  $\mathbb{J}_{n-1}$  which is closest to  $h_n(v)$  by an edge of length  $\frac{1}{2^n}$ . Then  $\dot{\Gamma}$  is homeomorphic to the constructed pseudo-arc, thus we get (i).



Let us denote by  $\Gamma_n$  the finite subgraph of  $\Gamma$  formed by all vertexes in  $\mathbb{J}_1, \mathbb{J}_2, \dots, \mathbb{J}_n$ . Note that there is a short map  $\Gamma_n \rightarrow \Gamma_{n-1}$  which is identity on  $\Gamma_{n-1}$ . It follows that for any path  $\alpha: [0, 1] \rightarrow \bar{\Gamma}$  we have that the total length of  $\alpha \setminus \dot{\Gamma}$  is at least  $|\alpha(0)\alpha(1)|_{\bar{\Gamma}}$ . Thus (ii) follows.

*Second part.* We construct a graph  $\Gamma^{(m)}$  to make  $\dot{\Gamma}^{(m)}$  homeomorphic to a product on  $m$  pseudo-arcs. Thus, the Lebesgue covering dimension of  $\dot{\Gamma}^{(m)}$  can be made arbitrary large.

We will do the case  $m = 2$ ; the others are analogous. The set of vertexes of  $\Gamma^{(2)}$  is disjointed union  $\sqcup_n (\text{Vert } \mathbb{J}_n \times \text{Vert } \mathbb{J}_n)$ , where  $\text{Vert } \mathbb{J}_n$  denotes the set of vertexes of  $\mathbb{J}_n$ . We connect two vertexes  $(x, y) \in \text{vert } \mathbb{J}_n \times \text{Vert } \mathbb{J}_n$  and  $(x', y') \in \text{vert } \mathbb{J}_k \times \text{Vert } \mathbb{J}_k$  iff the pairs  $(x, x')$  and  $(y, y')$  were connected in  $\Gamma$ ; the length of this edge must be maximum of lengths of edges  $xx'$  and  $yy'$  (we assume that a vertex is connected to it-self by an edge of length 0).

Clearly, there is a bijection  $\dot{\Gamma}^{(2)} \rightarrow \dot{\Gamma} \times \dot{\Gamma}$ . Note that there are two short coordinate projections  $\varsigma_1, \varsigma_2: \Gamma^{(2)} \rightarrow \Gamma$ . Thus for any path  $\alpha: [0, 1] \rightarrow \bar{\Gamma}^2$ , we have that total length of  $\alpha \setminus \dot{\Gamma}$  is at least as big as  $\max_i |\varsigma_i \circ \alpha(0)\varsigma_i \circ \alpha(1)|$ . That ensures that pulled back metric on  $\dot{\Gamma}^{(2)}$  is bi-Lipschitz to the product metric on  $\dot{\Gamma} \times \dot{\Gamma}$ .  $\square$

## 5 Comments and open questions

A length space  $\mathcal{M}$  is called Minkowski  $d$ -polyhedron if there is a finite triangulation of  $\mathcal{M}$  such that each simplex is isometric to a simplex in a Minkowski space. Correspondingly, a compact length space  $\mathcal{X}$  is called *pro-Minkowski space* of rank  $\leq d$  if it can be presented as an inverse limit of Minkowski  $d$ -polyhedra.

**5.1. Question.** *Is it true that any length space with Lebesgue's covering dimension  $\leq d$  is a pro-Minkowski space of rank  $d$ ?*

Or even more specific:

**5.2. Question.** *Is it true that any metric space which homeomorphic to a  $d$ -disk (say,  $d = 2$ ) is a pro-Minkowski space of rank  $d$ ?*

One can reformulate it in a more vague form: *Is there any essential difference between Finsler metric and general metric on  $n$ -manifold?* This question was asked by D. Burago; it was also original motivation for this paper (see also a related example [BIS, theorem 1]).

If one removes restriction on dimension, then the answer to the above question is YES. Namely, the following exercise can be solved by using Kuratowski embedding  $x \mapsto \text{dist}_x$ .

**5.3. Exercise.** *Show that any compact length space is an inverse limit of Minkowski polyhedra  $\mathcal{M}_n$  with  $\dim \mathcal{M}_n \rightarrow \infty$ .*

**5.4. Question.** *Is it true that any path isometry from a closed Euclidean ball to Euclidean space is an intrinsic isometry?*

## References

- [Akopyan] Akopyan, A. V., *A piecewise linear analogue of Nash–Kuiper theorem*, a preliminary version (in Russian) can be found on [www.moebiuscontest.ru](http://www.moebiuscontest.ru)
- [Akopyan–Tarasov] Akopyan, A. V.; Tarasov, A. S. *A constructive proof of Kirszbraun’s theorem*. (Russian) *Mat. Zametki* 84 (2008), no. 5, 781–784.
- [BIS] Burago, D.; Ivanov, S.; Shoenthal, D. *Two counterexamples in low dimensional length geometry*, *St.Petersburg Math. J.* Vol. 19 (2008), No. 1, Pages 33–43
- [BBI] Burago, D.; Burago, Yu.; Ivanov, S., *A course in metric geometry*. Graduate Studies in Mathematics, 33. American Mathematical Society, Providence, RI, 2001. xiv+415 pp. ISBN: 0-8218-2129-6
- [Brehm] Brehm, U., *Extensions of distance reducing mappings to piecewise congruent mappings on  $R^m$* . *J. Geom.* 16 (1981), no. 2, 187–193.
- [Federer] Federer, H., *Geometric Measure Theory*, Springer, 1969.
- [Lewis] Lewis, W. *The Pseudo-Arc*, *Bol. Soc. Mat. Mexicana* Volume 5 (1999), 25–77.
- [Gromov] Gromov, M., *Partial differential relations*, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)* 9, Springer-Verlag, Berlin, 1986, x+363,
- [Krat] Krat, S. *Approximation Problems in Length Geometry*, Thesis, 2005
- [Zalgaller] Zalgaller, V. A. Isometric imbedding of polyhedra. (Russian) *Dokl. Akad. Nauk SSSR* 123 1958 599–601.